# Motion of a Single Bead on a Bead Row: Theoretical Investigations

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Abstract. — The motion of a single bead on an inclined "line" made up of juxtaposed identical beads is analytically investigated. Initially, an entirely analytical method is presented in some detail: the problem is tractable with the usual analytical tools as long as any slip motion is ignored. Under these conditions, we show i) the existence of a steady regime within a certain range of line slopes and ii) that the mean velocity of the mobile bead may be calculated by induction. When compared to all the series of experiments (after the adjustment of the collisional parameter), this approach reflects well all the features displayed by experiments. The influence of slip is then numerically investigated, but, unfortunately, the unicity of solution to motion equations cannot be proved. Despite this uncertainty, energy considerations show that the main dissipation process is due to collisions. The analytical model is then compared to three alternative types of approach, namely a phenomenological approach, an heuristical model, and one based on molecular-dynamics model. The comparison throws light on the role of (solid) friction and collisions in granular materials: here, it is found that the macroscopic coefficient of friction (bulk resistance) is mainly due to geometrical characteristics and collisional effects.

# 1. Introduction

Granular media offer considerable scope for a variety of investigations, motivated to a large extent by industrial [1] and geotechnical [2] applications. Numerous geophysical flows also involve mixtures in granular form [3]. More recently, they have received increasing attention from physicists [4]. For instance, as typical of more complicated systems, the study of granular packing yields profound insights into disordered behaviour [5,6], while a sand heap may exemplify various phenomea, both unusual and full of complexities such as instability, non-linearity, phase change and so on [7–12].

In all these fields of investigations, a large number of experimental studies, theoretical models, and numerical simulations have endeavoured in the past decade to improve the understanding of granular media, as the rich literature testifies; nevertheless, our skill on fundamental mechanisms is often quite fragmented as compared to other complicated media (polymer melts,

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suspensions, etc.). This failing is mainly due to the concomitance of various bothersome phenomena, whether under static or dynamic conditions. For instance, in the language of rheologists, experimental results on granular flows are most often obscured by all the various causes of non-linearity (non-Newtonian flow, instability, thixotropy or phase transition) [13].

As a consequence, the most promising method for modelling granular flows currently consists in deducing the macroscopic behaviour on the basis of microscopic considerations. This approach, generally referred to as the microstructural approach, has been used in most new numerical [14-21] and theoretical models [22-27] and constitutes an alternative to other continuum theories, such as models based on thermodynamic considerations or phenomenological relations [28-31]. But microstructural models are clearly faced, in turn, with new issues, because, on the one hand, they shift the core of difficulty from a macroscopic level to a microscopic one and, on the other hand, they can lead to very complicated equations or numerical schemes. For example, when comparing the predictions of a microstructural theory with experimental data, it is rather tricky to assess the causes of discrepancy, because it may reflect either the lack of accuracy of the hypotheses used or the shortcoming of the approach (inadequate choice of interaction laws, average process, etc.). In this respect, the treatment of binary collisions within the kinetic theory and its subsequent developments is typical: initially, Newton's law for collisions between smooth elastic particles had been widely sufficient for impulse calculations within the kinetic theory of gases [32], and later, in the inchoate theories using this framework for rapid granular flows [26]; then, in attempts at improving kinetic models, more sophisticated assumptions have been used: making allowance for inelasticity [33], particle roughness [25] and change in spin [24] has clearly enhanced the accuracy of kinetic models, but in the meantime, it has greatly complicated formulation and multiplied the number of parameters required. Furthermore, it is not unusual to encounter models based on opposite assumptions and giving similar results: in the case of avalanches on a sand heap, numerical simulations carried out by Pöschel and Buchholtz and designed to test the role of friction have provided a concrete example of this troublesome point [34].

The basic idea of this paper is to study a problem which is as simple as possible, in the sense that the degree of freedom, the number of control parameters and the range of possible phenomena influencing the motion must be as low as possible and, in the meantime, it should be typical of problems encountered in granular material. On the other hand, it is suitable to consider a system which has already been studied experimentally and theoretically. With this view, the following system turns out to be suitable: it consists in a bead of radius r and mass m, in motion on a row of juxtaposed beads which have the same radius (R). This row constitutes a bumpy "line", which can be inclined at an angle  $(\delta)$  to the horizontal (see Fig. 1). Accordingly, two dimensionless control parameters characterize the geometry of the system, namely the ratio of radii  $\xi = r/R$  and the row slope  $\delta$ 

In a first section, the essential features displayed by experiments are summarized. In the following section, we focus on the theoretical determination of the average velocity of the mobile bead in a steady regime. To do so, we show that a completely analytical treatment is possible within the framework of rigid body mechanics. In Section 4, we discuss the assumption of pure rolling and its implications in motion equations and energy balance. The last section considers recent studies that deal with the prediction of the mean velocity of a spherical particle moving down an inclined bumpy line.

# 2. Description of the Physical System

Primarily, Jan *et al.* carried out experiments with a bumpy line, whose beads had the same diameter as the mobile bead ( $\xi = 1$ ) [35]. They tested several types of bead (steel or glass



Fig. 1. — Sketch of the motion of a bead on a bumpy line composed of juxtaposed beads.

beads and golf balls) and various values of radius (13.5, 21.9, 25.1, and 42.5 mm). The flume consisted in a bed layer, made up of beads closely packed together over a length of 3 m and surrounded by two smooth sidewalls. Later, Ristow, Riguidel, and Bideau extended Jan's results by investigating the influence of the radius ratio  $\xi$  [36]. The straight bed was made up of 400 steel beads of radius (R) 0.25 mm, confined in an L-shaped flume. Tests were conducted for several values of bed slope, the ratio  $\xi$  being comprised in the set {0.8, 1, 1.6, 2}.

Jan *et al.* showed that in all their tests, a steady motion of the mobile bead is achieved within a certain range of bed inclinations. In this case, they found that the mean velocity presumably does not depend on the density of the bead, and is proportional to the square root of the bead diameter.

Ristow's *et al.* results confirmed the existence of three regimes depending on the bed slope  $\delta$  and the regime boundaries were plotted in the form of a phase diagram (with  $\xi$  as control parameter):

- Regime A: the bead stops just after its departure regardless of its initial velocity.

- Regime B: the bead reaches an asymptotic velocity.

- Regime C: the beads lose the contact with bed beads and begins accelerated motion with jumps.

In Ristow's *et al.* results, an inflexion point exists for curves  $\xi = 1$  and 0.8 at the highest slope values, whereas such behaviour is not observed in Jan's data. We suggest that this change might be due, at least in part. to the transition from regime B to C, since this transition is presumably due to the occurrence of micro-jumps and accordingly implies an apparent increase in the mean velocity. It should also be noted that data corresponding to  $\xi = 1$  in Jan's and Ristow's experiments are similar but not identical: the steady motion began at 7.5° in Ristow's experiments instead of 11° in Jan's ones. This difference is probably due to the influence of experimental apparatus (effects of sidewalls, nature of material, etc.).

It is worth noticing that a parallel may be drawn between the existence of three phases of motion depending on slope and the three phases of motion in open channel for granular flows: according to Savage [1], a steady flow exists only for a wide range of channel slopes as soon as the slope exceeds a critical value (seemingly the angle of repose of the material); for slopes lower than this critical slope, an "immature sliding flow" (transitional regime with development

of stationary layers) takes place whereas the steady flow degenerates into a "splashing flow" (saltation of particles) when the channel slope is sufficiently greater than the critical slope. More puzzling is the fact that for channel flows or a single bead rolling on a bumpy line, the curves of mean velocity (as a function of slope) have an apparently similar form (see for instance [37]).

# 3. Analytical Study of Pure Rolling

3.1. PRINCIPLE OF THE APPROACH. — In this section, we seek to express analytically the mean velocity of the mobile bead and the boundaries of the steady regime. Here we propose a completely analytical treatment, which is based on the following hypotheses:

- Framework: calculations are performed within the framework of rigid body mechanics.
- *Kinematics:* the trajectory of the mobile bead is assumed to be inscribed in a single plane (one-dimensional motion) and the bead is assumed to roll without slipping (see below).
- *Dynamics:* the main forces acting on the mobile bead are the gravitational force and the effects due to friction and collisions (see below).
- Stability: the steady regime is implicitly assumed to exist and to be stable.

3.2. MODELLING OF CONTACT FORCES. — The mobile bead is assumed to be mainly subjected to the action of the gravitational and contact forces (with the bumpy line). Here, it seems natural to distinguish two possible contact configurations: the rolling contact (when the mobile bead rolls on the  $n^{\text{th}}$  bead of the bumpy line) and the collisional contact (when it goes from the  $n^{\text{th}}$  bead to the following).

3.2.1. Rolling Contact. — The contact between two smooth bodies generally gives rise, at the common interface, to a tangential force due to frictional processes, which opposes motion. According to Coulomb's law (see Eq. (33)), it is usual to distinguish two types of contact depending on the slipping velocity at the point of contact I: i) if this velocity is zero, the motion is referred to as "rolling without slipping" or "pure rolling" and the contact is called "sticking contact", ii) otherwise, the motion is referred to as "rolling with slipping", and slipping contact occurs. This takes place when the tangential force exceeds the limiting friction. Here, the coefficient of (solid) friction f, also called the Coulombic coefficient, is assumed to be constant regardless of the slipping velocity.

3.2.2. Collisional Contact. — A collision between two bodies is a very complicated event, whose main characteristic is the very brief duration of the contact with respect to usual time scales. Accordingly, the collisional process is assumed to be instantaneous and to provoke discontinuous changes of linear and angular velocities of each body. These changes are theoretically and experimentally investigated in the form of a "collisional law" between the initial and final relative velocity.

The colinear collision between two spheres (without spin) is the simplest configuration, for which the collisional law is Newton's law. This phenomenological relation has been theoretically justified by Hertz on the grounds that the dynamic process of deformations during a collision can be regarded as quasi-static and involves elastic deformations. The former assumption is expected to be relevant in so far as the contact duration is long enough to permit stress waves to traverse the length of both bodies many times (this condition is referred to as Love's criterion [38]). The latter assumption on elasticity yields a condition on the relative velocity before impact [39]. When the impact velocity exceeds this critical value, deformations become plastic and the coefficient of restitution (e) is introduced to account for energy lost in inelastic deformations. As primarily suggested by Tabor for colinear impact, it is possible to justify this phenomenological coefficient by considerations of elastoplastic deformations [38,39].

The case of elastic oblique impact or colinear impact between bodies with initial spin is more complicated: if the normal impulse can always be calculated by Hertz theory, the calculation of tangential traction arising in oblique impact is intricate owing to the occurrence of stick-slip contact [40,41]. Accordingly, a complete treatment is not directly possible within the framework of rigid body mechanics (as those proposed in [42–46]) and requires thorough examination of deformations in contact zones [47]. Continuum models, as those proposed by Maw *et al.* [48,49], Lim and Stronge [47] or Jaeger [50] have been validated by some experiments [48,51]; however, a rough phenomenological law is most often preferred to the detailed but intricate theory of Maw *et al.* [19,24]. In the case of two spheres, this consists in introducing both normal and tangential coefficients of restitution (respectively *e* and  $\beta_0$ ) [19,24,51].

In our particular case, an additional difficulty arises because the collision is not binary, but involves three beads. As a general formulation of this problem does not exist (as far as we know) and in absence of specific remarks in previous experimental works (on the bumpy line), we performed some tests with steel beads of diameter 1 cm: qualitative observations of our physical system showed that, apparently, no rebound occurred after impact between the mobile bead and a row bead. Accordingly, the (normal) coefficient of restitution should be zero and the collision should involve plastic deformations. In our context, it seems dubious to attribute the absence of rebound to plasticity: indeed according to Johnson's analysis on plasticity onset [38], plastic deformations occur for normal collision between spheres of the same radius as soon as the relative impact velocity exceeds a critical value, which is about 10 cm/s for steel beads. As mean velocities measured by Ristow or Jan et al. ranged from 4 to 14 cm/s, it is expected that collisions are elastic or slightly inelastic (e close to 1). Here, we suggest that a collision cannot be directly treated using Newton's law because, on the one hand, involving three beads, it is not binary, and, on the other hand, Love's criterion probably does not hold true, since the three beads involved are not isolated but on the contrary are connected to a network of juxtaposed beads.

In the following, we assume that, even for multiple collisions, the tangential component of the post-collisional velocity  $(V'_t)$  is a linear function of the normal and tangential components of the precollisional velocity V (this holds true for binary collisions [19,51]) as follows:

$$V'_{\rm t} = K(\alpha, f)V\tag{1}$$

where  $K(\alpha, f)$  is a coefficient, which we call the "collisional parameter", depending on control parameter  $\alpha$  and possibly on friction coefficient f (see below). K is a priori a unknown function.

3.2.3. Other Forces. — In this paper, we neglect the influence of air drag and flume sidewalls on bead motion:

- The magnitude of hydrodynamic forces acting on the mobile bead is expected to be low in comparison with the gravitational force (see [52] for a more complete analysis).
- The flume action is due to the fact that the spin vector does not necessarily have a fixed direction, and consequently a free bead never follows the top of the bead row [55] and is forced by the flume sides to move downwards and to roll on one side of the flume [36]; collisions with the flume sides are also possible. A possible treatment of the guidance

action of the flume consists in regarding the mobile bead as a cylinder rather than as a sphere. Consequently, we shall consider the gyration radius as a parameter of our model.

#### **3.3. MOTION EQUATIONS**

3.3.1. Degrees of Freedom: Notations. — Let us now consider motion on the bed bead (n). O and  $G_n$  denote respectively the mass center of the mobile and stationary beads. The studied system clearly possesses two degrees of freedom, since the motion can be broken down into rotation around the bead (n) and spin movement. The former motion is defined using the angle  $\theta$  between the vertical and the line joining the two mass centers, whereas the latter is defined via an angle  $\varphi$  with respect to a reference axis (see Fig. 1). In the following, we shall add an additional assumption on the motion, namely rolling without slipping, and thereby the degree of freedom of the system shall be reduced by one (see Eq. (6)). The motion is thus entirely determined by a single equation of variable  $\theta$ .

On the bead (n), rotation begins at an angle  $\theta_0$  and ends at an angle  $\theta_f$ . Obviously, both angles take the same value regardless of the bed bead and may be calculated by considering the mobile bead to be resting on two successive beads (see Fig. 1). To do so, we introduce the angle  $\alpha$  which is directly related to the roughness ratio  $\xi$ :

$$\alpha = \arcsin \frac{R}{R+r} = \arcsin \left(\frac{1}{1+\xi}\right) \text{ with } \xi = \frac{r}{R}, \tag{2}$$

From Figure 1, it is straightforward to find:  $\theta_0 = \delta - \alpha$  and  $\theta_f = \delta + \alpha$ . In the following, it is rather convenient to consider dimensionless variables, and consequently, we take respectively for time, length and force scales:  $1/\omega_0$ , (r+R), mg, where g denotes the gravity constant and  $\omega_0$  is defined as:

$$\omega_0^2 = \frac{g}{r+R}.$$
(3)

In the following, Greek capital letters along with tilded quantities refer to dimensionless parameters. When necessary, we will employ the new parameter  $\varepsilon$ , equivalent to  $\alpha$  or  $\xi$ :

$$\varepsilon = \frac{R+r}{r} = \frac{1+\xi}{\xi} = \frac{1}{1-\sin\alpha} \tag{4}$$

3.3.2. Relation Between the Mass Center and Spin Velocities. — We assume a sticking contact, so that the friction work is zero. Moreover, it is possible to link  $\varphi$  and  $\theta$ , or at least their derivatives, by writing that the slipping velocity at I is zero:

$$\mathbf{v}(\mathbf{I}) = \mathbf{v}(\mathbf{O}) - r(\dot{\varphi}\mathbf{e}_z) \times \mathbf{e}_{\mathbf{n}} = 0$$
(5)

This condition is equivalent to:

$$\dot{\Phi} = \frac{R+r}{r}\dot{\Theta} = \varepsilon\dot{\Theta} \tag{6}$$

3.3.3. Procedure of Calculation: Motion Equation. — We use an induction procedure in order to determine the mean velocity of the mobile bead. Initially, the bead is launched with a given velocity. The calculation is performed as follows (see Fig. 1):

- i) Consider that the bead is in motion on a given bed bead, which we shall assume to be the  $n^{\text{th}}$  bed bead encountered since the start of motion. From the energy balance equation, it is possible to infer the contact duration (for the bead n) and the velocity just before the collision with the following bead (n + 1).

- ii) The motion equation written for the previous bead is clearly valid for the bead (n+1), except that the initial velocity is altered and needs to be calculated with allowance for previous motion on the bead (n).
- iii) Finally, the calculation of the mean velocity is obtained by induction on n.

We are seeking to establish the motion equation for motion on the bead (n). This may be performed from the energy balance as follows. The kinetic energy  $E_k$  is the sum of two contributions: one part related to rotation energy (via  $\dot{\Theta}$ ) and the other to spin energy (via  $\dot{\Phi}$ ):

$$E_k = \frac{1}{2}m((r+R)^2\dot{\theta}^2 + \rho r^2\dot{\varphi}^2)$$
(7)

where  $\rho r^2$  is the gyration radius of the mobile body (with respect to its mass center). The energy balance implies that any change in kinetic energy is supplied by a variation in potential energy:

$$\frac{1}{2}m\left((r+R)^2 + \rho\varepsilon^2 r^2\right)(\dot{\theta}^2 - \theta_0^2) = mg(R+r)(\cos\theta_0 - \cos\theta)$$
(8)

where the angle  $\theta_0$  is chosen as reference angle and  $\dot{\theta}_{0,n}$  denotes the initial velocity on the  $n^{\text{th}}$  bead (at  $\theta = \theta_0$ ). The dimensionless expression of equation (8) writes:

$$\dot{\Theta}^2 - \dot{\Theta}^2_{0,n} = \frac{2}{1+\rho} (\cos\Theta_0 - \cos\Theta) \tag{9}$$

# 3.4. CALCULATION OF THE MEAN VELOCITY

3.4.1. Principle of the Calculation. — We define the (dimensionless) mean velocity of the mobile bead on the  $n^{\text{th}}$  bed bead as:

$$\overline{v}_n = \frac{2R}{R+r} \frac{1}{\tau_n} = \frac{2\sin\alpha}{\tau_n} \tag{10}$$

where  $\tau_n$  is the contact duration (on the  $n^{\text{th}}$  bead), which is obtained by integrating equation (9):

$$\tau_n = \int_{\Theta_0}^{\Theta_f} \frac{\mathrm{d}\theta}{\sqrt{\frac{2}{1+\rho}(\cos\Theta_0 - \cos\theta) + \dot{\Theta}_{0,n}^2}} \tag{11}$$

The appropriate change in variable enables us to write this integral in terms of elliptic functions [53]:

$$\gamma_n = \sqrt{\frac{4(1+\rho)}{2(1+\cos\Theta_0) + (1+\rho)\dot{\Theta}_{0,n}^2}},$$
(12)

$$k_n = \sqrt{\frac{1}{1+\rho}}\gamma_n, \tag{13}$$

$$\Omega_n(\Theta) = \arcsin \sqrt{(1 - \cos \Theta) \frac{2(1 + \cos \Theta_0) + (1 + \rho)\dot{\Theta}_{0,n}^2}{2(\cos \Theta_0 - \cos \Theta) + (1 + \rho)\dot{\Theta}_{0,n}^2}}$$
(14)

F is the elliptic integral of the first kind (Legendre's function). The equation (11) is then equivalent to the following form:

$$\tau_n = \gamma_n F(\Omega_n(\Theta_0), k_n) + \gamma_n F(\Omega_n(\Theta_f), k_n).$$
(15)

3.4.2. Induction Relations. — In order to calculate the limit of the sequence  $\overline{v}_n$ , we need to calculate the limits of the sequences  $\tau_n$  and  $\dot{\Theta}_{0,n}$ . To this end, let us display the induction relations linking these quantities. The final velocity (just before the impact with the following bead) is calculated from equation (9) at  $\Theta = \Theta_f$ :

$$\dot{\Theta}_{f,n} = \sqrt{\frac{2}{1+\rho} (\cos\Theta_0 - \cos\Theta_f) + \dot{\Theta}_{0,n}^2} \tag{16}$$

The initial velocity on the bead (n + 1) is deduced from the final velocity by way of the collisional law. Just before the impact, the normal and tangential components of the linear velocity are respectively  $\dot{\Theta}_{f,n} \sin 2\alpha$  and  $\dot{\Theta}_{f,n} \cos 2\alpha$ , the spin velocity is  $\varepsilon \dot{\Theta}_{f,n}$ . In the following, we shall examine two cases: i)  $K = \cos 2\alpha$  is referred to as "perfectly smooth collision" (continuity of velocity field: the tangential components of velocity before and after impact are equal) and ii)  $K = \cos(2\alpha) + \sin^2 \alpha \sin(2\alpha)/2$  is postulated taking into account multiple collisions and is calculated from data fitting (see Sect. 3.6.1.).

From equation (1) and (16), it may easily be shown that:

$$\dot{\Theta}_{f,n+1} = \mathcal{F}(\dot{\Theta}_{f,n}) \tag{17}$$

where the following function has been introduced:

$$\mathcal{F}(x) = \sqrt{x^2 K^2 + \frac{4}{1+\rho} \sin \alpha \sin \delta}$$
(18)

3.4.3. Limits of Sequence. — Obviously, taking into account that  $\mathcal{F}$  is an increasing function of x, the sequence is always ascending or descending depending on its initial value. In addition, as  $\mathcal{F}$  intersects the first bisector y = x, the sequence is bounded and converges to the intersection point, defined as  $\mathcal{L} = \mathcal{F}(\mathcal{L})$ ; solving this equation gives:

$$\mathcal{L} = \sqrt{\frac{4}{1+\rho} \frac{\sin \delta \sin \alpha}{1-K^2}}$$
(19)

We deduce directly the limit of  $\Theta_{0,n}$  and  $\tau_n$ , denoted respectively  $\mathcal{L}_0$  and  $\tau$ :

$$\mathcal{L}_0 = K\mathcal{L} \tag{20}$$

$$\tau = \gamma(F(\Omega_0, k) + F(\Omega_f, k))$$
(21)

where we used:

$$\gamma = \sqrt{\frac{2(1+\rho)}{\Lambda}},\tag{22}$$

$$k = \sqrt{\frac{1}{1+\rho}\gamma},\tag{23}$$

$$\Lambda = 1 + \cos(\delta - \alpha) + \frac{2K^2}{1 - K^2} \sin \alpha \sin \delta$$
(24)

$$\Omega_f = \arcsin\left[\sin\left(\frac{\delta+\alpha}{2}\right)\sqrt{\frac{\Lambda(1-K^2)}{\sin\delta\sin\alpha}}\right],\tag{25}$$

$$\Omega_0 = \arcsin\left[\left|\sin\frac{\delta-\alpha}{2}\right|\sqrt{\frac{\Lambda(1-K^2)}{K^2\sin\delta\sin\alpha}}\right]$$
(26)

Finally, we deduce the value of the mean velocity for a given slope  $\delta$  and a roughness parameter  $\alpha$ :

$$\overline{v}_{(\alpha,\delta)} = \frac{2\sin\alpha}{\gamma(F(\Omega_0,k) + F(\Omega_f,k))}$$
(27)

Note that this analytical expression may be approximated: as the elliptic modulus k is always about 1, it is possible to approximate the elliptic integral of the first kind as follows:

$$F(\Omega, k) \approx \ln(\tan \Omega + \cos^{-1} \Omega) = \operatorname{argsh}(\tan \Omega)$$
(28)  
$$k \to 1$$

Thus, equation (27) is approximated by:

$$\tilde{v}_{(\alpha,\delta)} = \frac{2\sin\alpha}{\gamma(\operatorname{argsh}(\tan\Omega_0) + \operatorname{argsh}(\tan\Omega_f))}$$
(29)

# 3.5. LIMITS OF STEADY REGIME

3.5.1. Upper Limit of Steady Regime. — When the bead velocity is too great, a jump occurs owing to the centrifugal effect; in other words, the normal component of the reaction force is reduced to zero. We estimate the upper limit (defined as the minimum bed slope sufficient to cause a jump) by writing that the normal component becomes zero at the final angle. As the body in motion undergoes only the action of gravity and reaction forces, we deduce from the momentum balance equation written in the mobile frame  $(\mathbf{e}_n, \mathbf{e}_t)$  the normal and tangential (dimensionless) components of the reaction force:

$$\tilde{N} = \cos \Theta - \dot{\Theta}^2 \tag{30a}$$

$$\tilde{T} = |\sin \Theta - \ddot{\Theta}| \tag{30b}$$

The jump condition is given by:  $\tilde{N} = 0$ . For a given roughness ratio, solving this equation and making use of equation (9) and equation (20) leads to the maximum angle for which a steady motion still exists:

$$\delta_{\rm d} = \arctan \frac{(1+\rho)(1-K^2)}{\tan \alpha (4+(1+\rho)(1-K^2))}$$
(31)

For the sake of brevity, the tilde over dimensionless quantities is omitted in the following.

3.5.2. Lower Boundary of Steady Regime. — When the bed slope is too low, motion becomes impossible: when the bead is falling in the trough between two successive beads, its kinetic energy is not sufficient to balance the decrease in potential energy and the bead stops. For the appraisal of the bed slope below which no motion is possible, let us consider the case where the mobile bead just reaches the top ( $\theta = 0$ ) of a bed bead with a zero velocity. Therefore, making use of equation (9) and (20), it may be shown that this situation corresponds to a bed slope  $\delta_{\rm m}$ .

$$\delta_{\rm m} = 2 \arctan\left(\frac{\sin\alpha}{1+\cos\alpha}\frac{1-K}{1+K}\right) \tag{32}$$

As a conclusion, we have to distinguish three regimes according to the bed slope  $\delta$ :

- $\delta < \delta_{\rm m}$  (Regime A): the bead stops just after its departure.
- $\delta_m < \delta < \delta_d$  (Regime B): the bead reaches an asymptotic velocity (given by Eq. (28)).
- $\delta_d < \delta$  (Regime C): the beads loses contact with the bed beads and begins a jumping motion.



Fig. 2. — Variation of the mean velocity in a steady regime with respect to the bed slope  $\delta$  and for various values of control parameter  $\xi$ . The broken line stands for  $\rho = 2/5$  (bead) whereas the full line represents the case  $\rho = 1/2$  (cylinder). The collisional parameter is chosen as  $K = \cos 2\alpha$ . Jan's *et al.* and Ristow's *et al.* data are also shown.

#### 3.6. Comparison with Experimental Data

3.6.1. Steady Regime. — In Figures 2 and 3, theoretical curves along with points corresponding to experimental measures of Ristow *et al.* and Jan *et al.* are reported against the bed slope  $\delta$  for four values of roughness parameter  $\xi$  and two values of the collisional parameter K.

In Figure 2, we have compared the trend given by Jan's and Ristow's data with our theoretical model in the case of continuity of velocity components, namely  $K = \cos 2\alpha$ . The curves of mean velocity corresponding to either  $\rho = 1/2$  (cylinder) or 2/5 (bead) are plotted: it turns out that theoretical results are slightly influenced by the gyration parameter except at steeper slopes and mean velocities, for which the relative difference between both theoretical curves reaches about 5%. A possible explanation lies in the fact that inertia effects are significant for larger velocities. When compared to experimental data, the trend of theoretical curves appears as correct but is not in complete agreement with experimental data. The relative difference ranges from 10 to 30%: it tends to decrease for larger values of  $\xi$  (for which it is about 10%) while it is greatest for smaller values of  $\xi$ , for which it exceeds 30%. As theoretical predictions are always lower than experimental results, we infer that the collisional parameter K actually should be slightly larger. This can be supported by the fact that a part of collisional energy, which is not restored in the form of a normal impulse (e = 0), could be released in the form of a tangential impulse. Accordingly, we have considered that the previous form  $K = \cos 2\alpha$  is to be corrected to account for multiple collisions and this correction on K is still a function of  $\alpha$ , since a multiple collision is presumably greatly influenced by the geometrical configuration of beads at impact.

From Ristow's *et al.* experiments, we have chosen a single value for each series of  $\xi$  and we have searched for the correct value of K for theoretical and experimental values of the corresponding mean velocities to match; we have thus established that the function  $K(\alpha) =$ 



Fig. 3. — Variation of the mean velocity in a steady regime with respect to bed slope  $\delta$  and for various values of control parameter  $\xi$  for the case  $K = \cos 2\alpha + 0.5 \sin^2 \alpha \sin 2\alpha$ . The gyration parameter is chosen as  $\rho = 1/2$ . Jan's *et al.* and Ristow's *et al.* data are also shown (for caption, see Fig. 2).

 $\cos(2\alpha) + \sin^2 \alpha \sin(2\alpha)/2$  fits correctly. As clearly shown in Figure 3, by way of this adjustement on four data, the relative difference between the trend given by Jan's and Ristow's experimental data and theoretical curves is less than 3% for four series  $\xi = 0.8$ , 1, 1.6, and 2, except for steepest slopes of the curve  $\xi = 0.8$ .

3.6.2. Motion Regimes. — It has been theoretically shown in Section 3.5 that a steady regime takes place provided that the bed slope is within a certain range, which only depends upon the control parameter  $\xi$ . Partitioning of this sort is in complete agreement with experimental observations: we have reported in Figure 4 the experimental data corresponding to phase changes along with the theoretical curves given by equations (31) and (32). As previously, we have to consider the influence of the collisional parameter K. On account of the slight influence of the gyration parameter, we have only considered bead motion ( $\rho = 2/5$ ).

First, let us consider the simplified case of  $K = \cos 2\alpha$ . As shown in Figure 4, theoretical results for the phase diagram are not entirely validated by experimental measurements: for the lower boundary, the difference reaches about 25% at larger values of  $\alpha$ , whereas for the upper limit, the relative error is greatest (30%) at lower values of  $\alpha$ .

Now, let us consider the case of the adjusted collisional parameter (see Fig. 4). Concerning the lower limit, theoretical predictions are in complete agreement with experimental data, but concerning the upper limit, analytical results only yield the trend, the relative difference ranging from 10 to 20%. This discrepancy might be regarded as a shortcoming of the model but it must actually be viewed more thoroughly by considering the uncertainty of experimental measurements coupled with the weakness of the model near the boundaries:

- It is experimentally hard to accurately estimate the transition between the regimes B and C. Ristow *et al.* determined the corresponding boundary by searching a sudden



Fig. 4. — Phase diagram versus the control parameter  $\alpha$  for both cases  $K = \cos 2\alpha$  (broken line) and  $K = \cos 2\alpha + 0.5 \sin^2 \alpha \sin 2\alpha$  (solid line). Ristow's *et al.* data are also shown: ( $\bigcirc$ ) stands for the lower limit; ( $\square$ ) represents the upper limit which we found from Ristow's *et al.* experiments (by considering that the inflexion point in their experimental curves corresponds to the limit between the regimes B and C) while ( $\blacksquare$ ) corresponds to the values given by the authors. The upper limit of the steady regime (B) is theoretically given by equation (31) and its lower limit by equation (32).

variation in the curve of the mean velocity plotted against the roughness parameter  $\xi$ . In this respect, it is not certain that the occurrence of micro-jumps is reflected by drastic changes in velocity curves; the inflexion point observed in some curves (see Sect. 2) corresponds perhaps to this phase transition.

- From a theoretical point of view, we can point out that the solution to the equation N = 0(giving the jump condition) depends to a large extent upon current velocity, whereas it is expected in the meantime that the actual velocity at  $\Theta = \Theta_f$  is greater than the one given by equation (9), because of slipping motion (see Sect. 4.4). Hence, the difference between experimental data and predictions of the analytical model can be expected as significant.

Finally, it is worth noticing that our analytical model predicts that no steady motion is possible for roughness ratios  $\xi$  of less than about 0.62; however, as we have neglected up to now the effect of slipping motion, this critical value is actually expected to be lower than 0.62 (see Sect. 4). Ristow's *et al.* experiments confirmed that no steady motion was observed for  $\xi$  lower than a critical value between 0.5 and 0.6.

# 4. Analysis Accounting for Slip Motion

All the previous considerations are based on the crucial assumption that no slipping takes place. We need to examine a *posteriori* the validity of this hypothesis. Then, we present motion equations accounting for slip motion, which unfortunately cannot be solved entirely by an analytical treatment. 4.1. RELIABILITY OF THE ASSUMPTION ON ROLLING. — The formulation of the motion equation (9) is based on the assumption that any slip at the point of contact cannot arise during motion. This holds provided that Coulomb's condition is fulfilled, namely when the tangential and normal components verify:

$$T \le fN \Rightarrow \mathbf{v}(\mathbf{I}) = 0 \tag{33}$$

otherwise slip occurs:

$$T = fN \text{ and } \mathbf{v}(\mathbf{I}) \neq 0 \tag{34}$$

In order to check whether Coulomb's condition is fulfilled, let us introduce the function  $H_n$  defined as the ratio of tangential to normal components:

$$H_n(\Theta) = \frac{T}{N} = \frac{\rho \sin \Theta}{(3+\rho)\cos\Theta - 2\cos(\delta-\alpha) - (1+\rho)\dot{\Theta}_{0,n}^2}$$
(35)

As soon as  $H_n$  reaches the value f, the rolling motion without slipping ceases; this arises at an angle:

$$\Theta_{G,n} = 2 \arctan \frac{-\rho + \sqrt{\rho^2 - f^2(\vartheta^2 - (3+\rho)^2)}}{f(3+\rho+\vartheta)}$$
(36)

with:  $\vartheta = 2\cos\Theta_0 + (1+\rho)\dot{\Theta}_{0,n}^2$ 

Slipping motion appears rather quickly on account of the low value of f (say 0.1) and therefore, the primary assumption of rolling without slipping could appear unsuitable. However, it will be shown later that the stick hypothesis yields a correct approximation of real motion anyway. To substantiate this statement, we shall try to determine the complete set of motion equations.

4.2. SET OF MOTION EQUATIONS. — The equation accounting for slipping motion is obtained with the help of Coulomb's law. Making use of equations (34) and (30), it is straightforward to deduce:

$$\sin \Theta - \ddot{\Theta} = f(\cos \Theta - \dot{\Theta}^2) \tag{37}$$

It is a non-linear second-order differential equation, rather uninteresting in this form. We expect that it may be transformed into a more suitable form, namely a first-order equation. With a view to decreasing the order, we shall try to express a possible solution to equation (37) in the following form:

$$\dot{\Theta}^2 = a e^{d\Theta} + b \sin\Theta + c \cos\Theta \tag{38}$$

For this "all-purpose" solution to exist, parameters b, c and d must fulfill:

$$\begin{cases} d = 2f\\ b = \frac{-6f}{1+4f^2}\\ c = \frac{4f^2 - 2}{1+4f^2} \end{cases}$$
(39)

These constants depend only on the nature of the material (but not on the gyration parameter), whereas the coefficient a depends on initial conditions (at slip onset), in other words by adding the subscript n:

$$a_n = (\dot{\Theta}_{G,n}^2 - b\sin\Theta_{G,n} - c\cos\Theta_{G,n})e^{-2f\Theta_{G,n}}$$
(40)

where  $\dot{\Theta}_{G,n}$  is the velocity at slip onset on the  $n^{\text{th}}$  bed bead.

Slip motion is characterized by a non-zero slip velocity at the point of contact. If this velocity falls to zero, then stick motion takes place again. It is thus necessary to calculate the (dimensionless) velocity at the point of contact:

$$\mathbf{v}(\mathbf{I}) = \mathbf{v}(\mathbf{O}) - \frac{\dot{\Phi}}{\varepsilon} (\mathbf{e}_z \times \mathbf{e}_n)$$
(41)

The spin velocity  $\dot{\Phi}$  is obtained from the balance equation for the moment of momentum (applied at the mass center O), namely:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{OI} \times \mathbf{v}(\mathbf{I})) = \mathbf{OI} \times (-T\mathbf{e}_{\mathrm{t}})$$
(42)

On account of equation (30b), it is easy to deduce  $\tilde{\Phi}$ :

$$\ddot{\Phi} = \frac{\varepsilon}{\rho} T \tag{43}$$

During slip motion, the spin velocity at any time t is obtained after integration from the slip onset, the initial condition being imposed by the stick motion (Eq. (6)):

$$\dot{\Phi}(t) = \varepsilon \dot{\Theta}_{G,n} + \frac{\varepsilon}{\rho} f \int_{t_{G,n}}^{t} (\cos \theta(u) - \dot{\theta}^2(u)) du$$
(44)

where  $t_{G,n}$  is the time corresponding to slip onset (t = 0 corresponds to the start of the motion on the  $n^{\text{th}}$  bead). The occurrence of slip motion considerably complicates the problem, since the spin velocity is now a function depending on previous motion. Slip motion exists as long as the velocity at the point of contact I fulfills:

$$\mathbf{v}(\mathbf{I}) \neq 0 \tag{45}$$

The set of equations governing motion is given by equation (9) and (38). The motion type (sticking or slipping contact) is controlled by a condition in the form of an inequality, namely Coulomb's law (33) for sticking motion and the condition on velocity at the point of contact (45) for slipping motion.

Unfortunately, in our context, it is not possible to determine a priori the actual type of motion as long as the collisional law is not correctly known. Indeed, the variation in spin velocity at impact entirely determines subsequent motion. It may be shown at least that two solutions to motion equations are possible: i) motion entirely governed by slip between  $\Theta_0$  and  $\Theta_f$  (referred to as "slip motion") and ii) a sequence of slip and stick motion (referred to as "stick/slip motion"). As the existence and the unicity of the solution to a (two-dimensional) problem with a friction process cannot be proved (see for instance [54]), it is not possible to choose the appropriate solution. In the following, we indicate how the two solutions are built.

4.3. SLIP MOTION. — In this case, we shall consider that the complete equations of motion can be reduced to equation (38) and (43). To calculate the mean velocity using the same method as previously (see Eq. (10)), we need to calculate the new expression of contact duration. For bed bead (n), the integration of the slip-motion equation makes it possible to infer the time function, which we express as an integral function of the initial and final angles:

$$\tau_n' = \int_{\Theta_0}^{\Theta_f} \frac{\mathrm{d}\theta}{G_n(\theta)} \tag{46}$$

where using (38):

$$G_n(\Theta) = \sqrt{a_n e^{2f(\Theta - \Theta_0)} + b\sin\Theta + c\cos\Theta}$$
(47)

and  $a_n$  denotes now:

$$a_n = \dot{\Theta}_{0,n}^2 - b\sin\Theta_0 - c\cos\Theta_0 \tag{48}$$

Assuming, once again, that the initial velocity on bead (n + 1) can be deduced from the final velocity on bead (n) via the collisional parameter K, the induction relation  $\mathcal{G}$  on the final velocity is:

$$\mathcal{G}(x) = \sqrt{(K^2 x^2 - b\sin\Theta_0 - c\cos\Theta_0)e^{4f\alpha} + b\sin\Theta_f + c\cos\Theta_f}$$
(49)

The limit of the final-velocity sequence is then the fixed point  $\mathcal{L}'$  of the function  $\mathcal{G}$ :

$$\mathcal{L}' = \sqrt{\frac{b\sin\Theta_f + c\cos\Theta_f - (b\sin\Theta_0 + \cos\Theta_0)e^{4f\alpha}}{1 - K^2 e^{4f\alpha}}}$$
(50)

As a consequence, the limit of  $(\tau'_n)$  is:

>

$$\tau' = \int_{\Theta_0}^{\Theta_f} \frac{\mathrm{d}\theta}{G(\theta)} \tag{51}$$

where:

$$G(\theta) = \sqrt{a_{\infty}} e^{2f(\theta - \Theta_0)} + b\sin\theta + c\cos\theta, \text{ and}$$
(52)

$$a_{\infty} = K^2 \mathcal{L}^2 - b \sin \Theta_0 - c \cos \Theta_0 \tag{53}$$

Unfortunately, there is no analytical solution to this integral expression, and we need to resort to either a numerical resolution or an approximation. We shall provide a clue for the latter choice; it is possible to give a fairly good approximation by expressing G as a function  $\hat{G}$  depending on circular *sine* and *cosine* functions. To this end, we point out that from a second order expansion, the exponential function can be approximated by:

$$e^x \approx 2 + \sin x - \cos x \tag{54}$$

By so doing, integration becomes possible in terms of elliptic functions. One obtains for  $\hat{G}$ :

$$\hat{G}(\Theta) = \sqrt{a'_{\infty} + b' \sin \Theta + c' \cos \Theta}$$
(55)

where:

$$a'_{\infty} = a_{\infty}(1+4f^2)e^{-2f\Theta_0}$$
 (56a)

$$b' = b + 2a_{\infty}f e^{-2f\Theta_0} \tag{56b}$$

$$c' = c - 4a_{\infty}f^2 e^{-2f\Theta_0}$$
 (56c)

$$\hat{\tau}' = \gamma'(F(\Xi(\Theta_0), k') - F(\Xi(\Theta), k'))$$
(57)

where we used:

$$k^{'2} = \frac{2p}{a_{cr}' + p},$$
 (58a)

$$\gamma' = \frac{2}{\sqrt{a'_{\infty} + p}} \tag{58b}$$

$$p = \sqrt{b'^2 + c'^2},$$
 (58c)

along with the variable change:

$$\Xi(\mathcal{V}) = \arcsin\sqrt{\frac{p - b'\sin(\mathcal{V}) - c'\cos(\mathcal{V})}{2p}}.$$
(59)

Making use of equation (10) and (57) or (51) leads to the new expression of mean velocity:

$$\overline{v}'_{(\alpha,\delta)} = \frac{2\sin\alpha}{\tau'} \approx \frac{2\sin\alpha}{\hat{\tau}'} \tag{60}$$

This calculation is reliable as long as the condition (45) holds true. If  $\Phi_{0,n}$  denotes the initial spin velocity for the  $n^{\text{th}}$  bead, then the final spin velocity is on the basis of equation (44):

$$\dot{\Phi}_{f,n} = \dot{\Phi}_{0,n} + f \frac{\varepsilon}{\rho} \int_0^{\tau'} (\cos \Theta(t) - \dot{\Theta}^2(t)) dt$$
(61)

where  $\Theta(t)$  is the solution to motion equation (38). An approximated solution is obtained using the above techniques (Eq. (54)):

$$\hat{\Theta}(t) = \pi - 2 \arctan \frac{b' + 2p \operatorname{sn}(t/\gamma') \operatorname{cn}(t/\gamma')}{p(1 - 2\operatorname{sn}^2(t/\gamma')) - c'}$$
(62)

where "sn" and "cn" are, respectively, the Jacobian sine and cosine. As before, we shall suppose that the spin velocity just after the collision  $\Phi_{0,n+1}$  is linked to the pre-collisional velocity  $\dot{\Phi}_{f,n}$  via a collisional parameter L, which should be a function of  $\alpha$ , f and  $\dot{\Theta}_{f,n}$  (i.e.  $\mathcal{L}'$ ). Accordingly, the asymptotic value of the initial spin velocity is:

$$\dot{\Phi}_0 = \frac{f\varepsilon L(\alpha, f, \mathcal{L}')}{\rho(1 - L(\alpha, f, \mathcal{L}'))} \int_0^{\tau'} (\cos\Theta(t) - \dot{\Theta}^2(t)) dt$$
(63)

As the function L cannot either be known theoretically or fitted experimentally, it is not possible to check the condition (45).

4.4. STICK/SLIP MOTION. — An alternative method exists for building a solution to the complete equations of motion. It consists in noting that sticking motion is possible within the range of slopes  $[-\Theta_{G,n}, \Theta_{G,n}]$ , given by equation (36). Outside this range, slipping motion takes place. The sequence of motion for the  $n^{\text{th}}$  bead depends on the value of the initial angle  $\Theta_0$ : if  $\Theta_0 < \Theta_{G,n}$ , then the motion includes three successive phases (slip, stick and slip again); otherwise it encompasses two stages (stick then slip). As  $\Theta_{G,n}$  has a value which is always close to 15° (see Eq. (36)) and on account of the values of  $\Theta_0$ , we shall consider in the following only the latter sequence of motion. Otherwise, the calculation is more complicated (among other things, the Eq. (36) is no longer true and must be replaced by an induction calculation); moreover, it may be shown this complete calculation yields only slightly different results.

In this perspective, we shall calculate the duration of each motion phase in order to calculate the contact duration on the  $n^{\text{th}}$  bead. Here,  $t_{G,n}$  denotes the duration of the stick phase on the bead (n), using equation (15):

$$t_{G,n} = \gamma_n(F(\Omega_n(\Theta_0), k_n) + F(\Omega_n(\Theta_{G,n}), k_n))$$
(64)

while, the total duration writes:

$$\tau_n'' = t_{G,n} + \int_{\Theta_{G,n}}^{\Theta_f} \frac{\mathrm{d}\theta}{H_n(\theta)} \tag{65}$$

where:

$$H_n(\Theta) = \sqrt{a_n'' e^{2f(\Theta - \Theta_{G,n})} + b\sin\Theta + c\cos\Theta}$$
(66)

(which can be approximated as previously for G) and  $a''_n$  denotes:

$$a_n'' = \frac{2}{1+\rho} \cos \Theta_0 + \dot{\Theta}_0^2 - b \sin \Theta_{G,n} - (c + \frac{2}{1+\rho}) \cos \Theta_{G,n}$$
(67)

As previously, showing the convergence of the sequence  $\overline{v}_n$  is tantamount to demonstrating the convergence of the final velocity on bead (n), defined from an induction relation  $\mathcal{H}$ . Even if the limit of this sequence cannot be found analytically, it is easy on account of equation (36), (38) and (67), to show that a fixed point  $\mathcal{L}''$  exists since the induction function  $\mathcal{H}$  asymptotically behaves as  $\mathcal{H}(x) \propto K e^{2f\alpha} x$ , thus the corresponding curve intersects the line y = x, provided that  $K e^{2f\alpha}$  is lower than 1. The mean velocity is directly deduced:

$$\overline{v}_{(\alpha,\delta)}^{\prime\prime} = \frac{2\sin\alpha}{\tau^{\prime\prime}} \tag{68}$$

4.5. COMPARISON OF SOLUTIONS. — As the collisional law is not correctly known, any serious prediction of the previous equations (60) and (68) needs an adjustment of the collisional parameter for each motion sequence; consequently, the comparison (in terms of mean velocity) between both solutions will be biased, since the value of K would be twofold. Here, we prefer to only consider the following problem: for any set  $(\alpha, \delta)$ , we are seeking the adequate collisional parameter K from experimental data. This has been numerically achieved by searching for a root to the equation:

$$\overline{v}_{\rm th}(K,\alpha,\delta) = \overline{v}_{\rm exp.}(\alpha,\delta) \tag{69}$$

where  $\overline{v}_{th}$  is the theoretical mean velocity (given by Eq. (60) or (68)) whereas  $\overline{v}_{exp}$  is the mean velocity experimentally measured for the set  $(\alpha, \delta)$ .

Corresponding values of K are plotted in Figure 5 and 6 for each motion sequence. In Figure 5, the stick-slip motion implies that K tends towards  $\cos(2\alpha)$ . The difference between Kand  $\cos(2\alpha)$  is probably due to the first phase of slip which has been ignored. The interpretation of this asymptotic behaviour is clear: the collision entails no discontinuity in tangential velocity (of mass center). In this perspective, this implies that the previous adjustment of K (see Sect. 3.6.) could be an artefact: actually, K would represent the influence of the slip phase rather than an effect of multiple collisions. In Figure 6, motion including only one slip phase means that K should be a decreasing function of the final velocity, but we have failed to yield any interpretation to this behaviour. In the light of these comments, the sequence of stick-slip motion appears to us to be the most physical solution.

It is also of interest to compare theoretical predictions with and without a slip phase. As an example of errors caused by the rolling assumption, we have plotted in Figure 7 the variation of local velocity with respect to the angle of position  $\Theta$  by considering either a (pure) rolling or slipping motion. For  $\xi = 2$  and  $\delta = 6^{\circ}$ , the relative difference between both solutions is about 6% whereas it reaches 15% for  $\xi = 1$  ( $\delta = 12^{\circ}$ ). The pure-rolling assumption turns out to be all the more adequate since the parameter  $\xi$  is large. This is readily explained by the fact that the angular trajectory is  $2\alpha$ , and consequently, on account of equation (2) it decreases when  $\xi$  increases. This fairly correct agreement between both predictions explains the reason why pure rolling motion is a good approximation of real motion (in the sense that the complete solution encompassing phases of sticking and slipping is thought to be a correct representation of reality).



Fig. 5. — Variation of K as a function of the final velocity (in a steady state) calculated from equation (69) in the case of a stick/slip solution, as given by equation (68) for various values of the ratio  $\xi$ . K is calculated from Ristow's data for  $\xi = 0.8, 1, 1.6, \text{ and } 2$  and from Jan's model for  $\xi = 1$ . The four lines indicate the value cos  $2\alpha$  for each series, which is also shown in italics.



Fig. 6. — Variation of K as a function of the final velocity (in a steady state) calculated from equation (69) in the case of a slip solution, as given by equation (60) for various values of the ratio  $\xi$ . K is calculated from Ristow's data for  $\xi = 0.8, 1, 1.6, \text{ and } 2$ .



Fig. 7. — Comparison of the velocity calculated with either the assumption of pure rolling motion (broken line), slip motion (discontinuous line) or stick/slip motion (continuous line), given by equation (9) or (38). The local velocity is represented for two particular cases: ( $\xi = 2$ ,  $\delta = 5.73^{\circ}$ ) and ( $\xi = 1$ ,  $\delta = 11.53^{\circ}$ ). The friction coefficient is f = 0.1 and  $\rho = 2/5$ .

4.6. DISCUSSION OF BOTH MODELS. — The above analysis has focused on two essential developments, depending on the account of the slip phase:

In the former development, we have considered a motion without slipping and we have taken advantage of the fact that a rolling assumption offers a significant gain in mathematical formulation and provides an entirely analytical solution. Although this hypothesis turned out to be crude, the consistency of our analytical model has been proved from comparisons with experimental data: it was able to correctly predict the mean velocity of the mobile bead as a function of control parameters  $\delta$  and  $\xi$  as well as the existence of three motion regimes. In addition, this method provided further indications on our physical system:

- First, it is worth noticing that steady motion is possible; in other words, from a macroscopic point of view, a "force" balances the downward component of gravity force. It is quite easy to convince oneself that this force is only due to energy loss during collisions: in this respect, we should consider the part of energy lost in collisions and the gain in potential energy (over a path corresponding to the distance between two successive collisions). The corresponding ratio may be expressed as:

$$\frac{\Delta E_{\rm k}}{E_{\rm p}} = \frac{\frac{1}{2}(1+\rho)(K^2-1)\mathcal{L}^2}{2\sin\alpha\sin\delta}$$
(70)

where  $\Delta E_k$  is the variation in total kinetic energy. In this balance about 30% of the kinetic energy is due to the rotational motion.

Using equation (19), we find naturally:

$$\frac{\Delta E_k}{E_p} = -1 \tag{71}$$

- Secondly, the analytical approach has shown that a suitable determination of the collisional law (via the collisional parameter K) is crucial for the correct calculation of mean velocities and

that the gyration parameter does not apparently play a significant role in the mean motion. The collisional parameter cannot be predicted by current modelling of binary collisions and, on the contrary, it is thought to pertain to energy transfer (from normal to tangential direction) in multiple collisions.

The second development investigates the occurrence and influence of a slip phase in the motion of the bead. As matters stand, it has been shown that the unicity of the solution to the motion equations cannot be proved. Furthermore, with allowance for friction, the energy conservation equation (70) becomes:

$$\frac{\Delta E_k}{E_p} = \frac{\frac{1}{2}(1+\rho)(K^2-1)\mathcal{L}''^2}{4\sin\alpha\sin\delta}$$
(72)

Unfortunately, this ratio cannot be calculated analytically, since no analytical expression for  $\mathcal{L}''$  and  $\dot{\Phi}$  exists. Nevertheless, considering only dissipations due to translational kinetic energy and taking for K the values given by equation (69), we find from Ristow's data that the ratio of translational kinetic energy to potential energy always ranges from 74% to 94%. Thus, even taking into account slip motion, the collisional process constitutes the main dissipation compared to the frictional effect within the range of Jan's and Ristow's experiments.

### 5. Survey of Alternative Approaches

In the following, a thorough analysis of three model types is presented: Jan's model is phenomenological in the sense that his analysis is intended to represent experimental data as simply as possible by considering energy balance [35]; Jaeger *et al.* proposed a more sophisticated model supported by heuristic arguments [55], whereas Ristow *et al.* performed a series of numerical simulations based on methods used in molecular dynamics [36].

# 5.1. PHENOMENOLOGICAL APPROACH

5.1.1. Principle. — In order to estimate the mean velocity (space average), Jan *et al.* considered, in a steady regime, the energy balance over a length 2R between two successive collisions [35]. The gain in potential energy must be lost in friction and inelastic dissipations due to collisions:

$$mg2R\sin\delta = m\lambda\overline{v}^2 + \mu mg2R\cos\delta \tag{73}$$

where  $\mu$  is a friction coefficient, that the authors called "bulk resistance coefficient", and  $\lambda$  a coefficient pertaining to collisional dissipations. From this balance, the (dimensionless) mean velocity is easily inferred:

$$\overline{v} = \sqrt{\frac{2\sin\alpha}{\lambda}(\sin\delta - \mu\cos\delta)}$$
(74)

5.1.2. Comparison with Experimental Data and Discussion. — In Figure 8, we have plotted  $\overline{v}^2/\cos\phi$  as a function of  $\tan \delta$ . According to their authors, the model successfully captured the trend for the data series  $\xi = 1$ . The friction coefficient is found to be close to 0.13, which is the usual order of magnitude for the Coulomb coefficient of steel. However, without resorting to parameter fitting, the model is unable to give correct values for other values of  $\xi$ . This difficulty may be circumvented by fitting both parameters  $\lambda$  and  $\mu$  from experimental data. From Ristow's series, we found that the more appropriate fitting is respectively for  $\lambda$  and  $\mu$  (as functions of  $\alpha$ ):

$$\lambda = 0.063 \mathrm{e}^{4.96 \sin \alpha} \tag{75}$$

$$\mu = 0.0025 e^{7.46 \sin \alpha} \tag{76}$$



Fig. 8. — Variation of  $v^2/\cos \delta$  as a function of  $\tan \delta$ . The curves given by Jan's model are plotted with allowance for the fitting (75) and (76). Jan's *et al.* and Ristow's *et al.* data are also shown.

By way of this adjustment, Jan's model comes closer to describing the observed features. Nevertheless, we can point out that it is tricky to propose an interpretation of the coefficients introduced in Jan's model. Indeed, the model is based upon frictional and collisional dissipations via  $\lambda$  and  $\mu$ :

- i) With regard to collisions, the authors related the energy loss to the mean velocity, whereas it actually depends on the final velocity along with spin velocity (at impact). As the mean and final velocities are not closely related, the two expressions (73) and (70) are not equivalent. Here the ratio of collisional loss to potential energy is written as:

$$\frac{\Delta E_{\mathbf{k}}}{E_{\mathbf{p}}} = \frac{\lambda \overline{v}^2}{2\sin\alpha\sin\delta} \tag{77}$$

It ranges from 0.24 to 0.78, which differs considerably from the previous range for stick / slip motion [0.74, 0.94] or the value for stick motion (1).

- ii) With respect to the friction coefficient  $\mu$ , our own results cast some doubt on the interpretation as a solid friction coefficient. By extending the author's hypothesis, we also suggest that  $\mu$  is the expression of macroscopic ("bulk") friction, due to a large extent to geometrical constraints and collisional effects. To show that, we have compared both analytical expressions given by equation (74) and (27). It is straightforward to show that:

$$\mu = \lim_{v \to 0} \tan \delta \tag{78}$$

Thus:

$$\mu = \frac{(1 - K^2)\sin\alpha}{(1 + K^2)\cos\alpha + 2K}$$
(79)

The equation (79) yields friction coefficient values accurate to within to 15%, when compared to the fitted values (76). The parallel may illuminate the notion of macroscopic friction, which is used, for example, with the internal friction angle or the repose angle for a sand heap. Pöschel and Buchholtz concluded, in a similar way, that friction phenomena are presumably more attributable to particle geometry than solid friction (Coulomb's law) [34], while recent experiments carried out by Abriak on a triaxial and numerical simulations performed by Cambou confirmed the slight dependence of the friction angle on the Coulomb coefficient [23, 56].

#### 5.2. HEURISTIC MODEL

5.2.1. Principle. — In order to explain the hysteresis phenomena commonly observed in granular avalanches, Jaeger *et al.* proposed a model, based on heuristic considerations, which expressed the mean frictional force acting on a single particle within a bulk [55]. This force can be separated into two contributions: one part related to the collisional process ( $\mathcal{R}_c$ ) and another to dissipations due to friction and variations in potential energy ( $\mathcal{R}_p$ ). After examining the asymptotic behaviour of these terms at low and large shear-rates, the authors proposed the following extrapolated form:

$$\mathcal{R} = \mathcal{R}_{\rm c} + \mathcal{R}_{\rm p} = (a_3 \mathrm{v}^2 + \frac{a_1}{1 + a_2 \mathrm{v}^2}) mg \cos \delta, \tag{80}$$

where the dimensionless velocity is used:

$$\mathbf{v} = \frac{v}{\sqrt{2gr\cos\delta}} \tag{81}$$

and where  $a_2$  and  $a_3$  are constants pertaining to particle geometry and energy loss during collisions, while  $a_1$  is a constant rendering the depth of depression as well as the solid friction;  $a_1$  is related to the angle of maximum stability for a sand heap. The previous expression exhibits a minimum provided that the coefficients fulfill:

$$a_1 a_2 > a_3 \tag{82}$$

From the form proposed by Jaeger *et al.* (Eq. (80)), we found that, in our context, the steady-state condition implies:

$$\tan \delta = a_3 \mathbf{v}^2 + \frac{a_1}{1 + a_2 \mathbf{v}^2} \tag{83}$$

By performing the variable change:  $X = v^2$ , this expression is equivalent to a second-order polynomial:

$$a_2 a_3 X^2 + X(a_3 - a_2 \tan \delta) + a_1 - \tan \delta = 0$$
(84)

Real square roots of this equations exist if  $\Delta \ge 0$ , where  $\Delta$  is :

$$\Delta = (a_3 + a_2 \tan \delta)^2 - 4a_1 a_2 a_3$$
  

$$\Delta = (a_3 + a_2 \tan \delta - 2\sqrt{a_1 a_2 a_3})(a_3 + a_2 \tan \delta + 2\sqrt{a_1 a_2 a_3})$$
(85)

The existence of real square roots and how many may be found, depend on the sign of  $\Delta$ . All the possible cases are summarized in Table I.

Note that if we have:

$$2\sqrt{\frac{a_1a_2}{a_3}} < 1$$
 (86)

		$a_3 > a_2 \tan \delta$	$a_3 < a_2 \tan \delta$
	$\tan \delta < a_1$	no positive	two positive
$\tan > \delta \frac{a_3}{a_2} (2\sqrt{\frac{a_1a_2}{a_3}} - 1)$		square root	square roots
	$\tan \delta > a_1$	one positive	one positive
		square root	square roots
	$\tan \delta < a_1$	no positive	one positive
$\tan \delta = \frac{a_3}{a_2} (2\sqrt{\frac{a_1 a_2}{a_3}} - 1)$		square root	square root
	$\tan \delta > a_1$	one positive	one positive
		square root	square root
		no solution	no solution
$\tan \delta < \frac{\beta}{\alpha} (2\sqrt{\frac{a\alpha}{\beta}} - 1)$			

Table I. — Several solutions to equation (84) depending to slope and parameters  $a_2$ ,  $a_3$ .

then motion is always possible regardless of the bed slope, which is unphysical. Accordingly, we shall assume that such an inequality is never fulfilled. We can deduce the expression of mean velocity, in so far as a positive root exists:

$$\overline{\mathbf{v}} = \sqrt{\frac{1}{a_2\varepsilon}(\sin\delta - \chi(\delta)\cos\delta)} \tag{87}$$

where we used the function  $\chi$  of slope  $\delta$ :

$$\chi(\delta) = \frac{1}{a_2} (a_3 - \sqrt{(a_3 + a_2 \tan \delta)^2 - 4a_1 a_2 a_3})$$
(88)

5.2.2. Comparison with Experimental Data and Discussion. — The expression of the mean velocity is similar to the one found in Jan's model, except that the friction coefficient is replaced by the function  $\chi(\delta)$ , which also depends on  $a_1$ ,  $a_2$ , and  $a_3$ . We have attempted to fit these parameters using experimental data. To that end, we express the mean velocity as a function of  $\tan \delta$ :

$$\overline{v}^2 = \frac{a_2 \tan \delta - a_3 + \sqrt{(a_3 + a_2 \tan \delta)^2 - 4a_1 a_2 a_3}}{a_1 a_2 a_3 \sqrt{1 + \tan^2 \delta}}$$
(89)

By taking several possible combinations of data for the same series, we unfortunately found that these parameters vary significantly as a function of the slope depending on the set chosen from the data: for example, in the case of  $\xi = 1$ ,  $a_1 = 0.131$ ;  $6.11 < a_2 < 20.83$  and  $1.2 < a_3 < 1.5$ . We nevertheless found a rough set of parameters adjusted from each series of  $\xi$ . The coefficient  $a_1$  was always chosen to be equal to the minimum slope necessary for steady motion. Corresponding fitted curves are plotted in Figure 9. The global trend is correct, but fails to represent details.

In addition, Jaeger's model predicts three regimes according to the bed slope, as summarized in the previous table: a steady regime occurs when  $\tan \delta$  is in excess of  $a_1$ , whereas a bi-stable regime arises if the slope is between:

$$a_1 > \tan \delta > \frac{a_3}{a_2} \left( 2\sqrt{\frac{a_1 a_2}{a_3}} - 1 \right)$$
 (90)



Fig. 9. — Adjustment of Jaeger's model from Ristow's experiments. We found:  $(\xi = 0.8) a_1 = 0.19$ ;  $a_2 = 3.83$ ;  $a_3 = 1.28/(\xi = 1) a_1 = 0.131$ ;  $a_2 = 18.57$ ;  $a_3 = 1.41/(\xi = 1.6) a_1 = 0.06$ ;  $a_2 = 20.71$ ;  $a_3 = 0.8987/(\xi = 2) a_1 = 0.042$ ;  $a_2 = 27.67$ ;  $a_3 = 0.78$ . Ristow's *et al.* data are also shown (see Fig. 8 for caption).



Fig. 10. — Predictions of Ristow's model after [36]. Jan's et al. and Ristow's et al. data are also shown.

This condition means that two asymptotic values for velocity exist and consequently, the system depends on initial conditions. Such a regime conflicts with our own analytical results, since it has been shown that the mean-velocity sequence converges towards the same limit regardless of its initial value and no clear experimental evidence can support this result.

#### 5.3. MOLECULAR-DYNAMICS MODEL

5.3.1. Principle. — Using methods for modelling assemblies of molecules (molecular dynamics), Ristow *et al.* performed numerical simulations of the motion of a single bead on a bumpy line [36]. This deterministic approach, commonly used in most numerical investigations on granular materials, proves to be fruitful because it makes it to estimate flow properties (velocity, density, and so on) inside the material without disturbing the flow field contrary to laboratory experiments (see the reviews [57,58]). Within this framework, the force  $\Re$  results from collisional interactions between particles:

$$\boldsymbol{\Re} = \begin{pmatrix} -k_{\mathrm{n}}(r_{\iota} + r_{\jmath} - \mathbf{e}_{\mathrm{n}}(\mathbf{r}_{\iota} - \mathbf{r}_{\jmath}))^{\beta} - \gamma_{\mathrm{n}} m_{\mathrm{eff}} \mathbf{e}_{\mathrm{n}}(\dot{\mathbf{r}}_{\iota} - \dot{\mathbf{r}}_{\jmath}) \\ -\gamma_{\mathrm{s}} m_{\mathrm{eff}}(\mathbf{e}_{\mathrm{t}}(\dot{\mathbf{r}}_{\iota} - \dot{\mathbf{r}}_{\jmath}) + r_{\iota} \omega_{\iota} + r_{\jmath} \omega_{\jmath}) \end{pmatrix}$$
(91)

where  $\mathbf{r}_i$  denotes the position vector of the mass center of  $i^{\text{th}}$  bead with respect to a fixed point O,  $\omega_i$  is the spin velocity, and  $m_{\text{eff}}$  the reduced mass:

$$m_{\rm eff} = \frac{m_i m_j}{m_i + m_j} \tag{92}$$

The constant  $k_n$  is Young's modulus,  $\gamma_n$  the damping coefficient in the normal direction,  $\gamma_s$ and  $\beta$  are constants accounting, respectively, for the transfer of translational kinetic energy into rotational energy and for elastic deformation. In their tests, the authors took the following values:  $k_n = 107$  Pa,  $\beta = 1.5$ ,  $\gamma_n = 1000$  Hz (e = 0.7),  $\gamma_s = 100$  Hz. It should be noted that they did not take solid friction into account. We should point out that the physical parameters used by Ristow *et al.* are different from usual values: for instance, Young's modulus is  $2 \times 10^4$ less than the common value for steel and a non-zero coefficient of restitution should result in a rebound, which has not been experimentally observed.

5.3.2. Comparison with Experimental Data and Discussion. — The predictions of the numerical model are plotted in Figure 10 and can be compared with the experimental data. This numerical model yields correct trends for mean velocities and changes in phase; the relative difference between experimental data and numerical results does not exceed 40%.

### 6. Conclusion

A simple physical system, consisting in the motion of a single bead on an inclined bumpy line, has been studied using various approaches. Our analytical approach, complemented by numerical analysis, has raised several crucial points: i) a steady regime exists within a certain range of bed slopes (whose limits depend on the radius ratio  $\xi$  and the collisional parameter K), because the gain in potential energy is mainly lost in collisions, and to a lesser degree in friction; ii) no rebound occurs in the regime B; iii) the chief parameter is the collisional parameter K whereas the gyration parameter  $\rho$  plays a minor role. Moreover, the assumption of pure rolling makes it possible to approximate the major features of the physical system. The agreement becomes very good when the value of K is adjusted using experimental data; in this perspective, K is expected to be due to collisional effects. In fact, slipping effects seem to be the actual cause, as numerical tests (accounting for slipping motion) show more accurately. This simple example provides clear evidence that the discrepancy between experimental and theoretical results can be tricky to analyse.

In parallel, other approaches have been tested. The simple model of Jan *et al.* requires an appropriate adjustment of frictional and collisional parameters to correctly predict the mean velocity. The coefficient of macroscopic friction is found to be mainly a function of collisional

and geometrical characteristics, depending only slightly on solid friction. This relation can shed light on the notion of the internal friction angle used in soil mechanics. Although intermittent regimes have been observed for flows on sand heaps or down open channels, there is no experimental evidence that such a regime exists for our physical system, and consequently, there are few reasons in our context to use the sophisticated model of Jaeger *et al.*, which requires the adjustment of three parameters. The trend given by the numerical model of Ristow *et al.* is roughly correct, but is based on the use of unphysical values for material parameters.

A central point in the development of the microstructural model has not been addressed in this paper: microstructural models are generally concerned with systems composed of a very large amount of particles, and it is precisely this large number which makes it possible (for instance, in the kinetic theory of gases) to describe the mean behaviour. In this respect, the low accuracy of the molecular-dynamics model of Ristow *et al.* is not surprising in the case of a single bead rolling on a bumpy line, but is not proved for larger systems.

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#### References

- [1] Savage S.B., Adv. Appl. Mech. 24 (1984) 289-366.
- [2] Wieghardt K., Ann. Rev. Fluid Mech. 7 (1975) 89-114.
- [3] Savage S.B., Flow of granular materials, Theoretical and Applied Mechanics, P. Germain, J.-M. Piau and D. Caillerie, Eds. (Elsevier, Amstersdam, 1989) pp. 241-266.
- [4] Jaeger H.M. and Nagel S.R., Science 255 (1992) 1523-1531.
- [5] Bideau D. and Dodds J., Physics of Granular Media (Nova Science Publishers, New York, 1991) pp. 434.
- [6] Bideau D. and Hansen A., Disorder and Granular Media (North-Holland, Amsterdam, 1993) pp. 323.
- [7] Evesque P. and Rajchenbach J., Phys. Rev. Lett. 62 (1989) 44-46.
- [8] Evesque E. et al., Phys. Rev. E 47 (1993) 2326-2332.
- [9] Rajchenbach J. and Evesque P., C.R. Acad. Sci. Paris 307 (1988) 1-4.
- [10] Jaeger H.M., Liu C.H. and Nagel S.R., Phys. Rev. A 43 (1991) 7091-7092.
- [11] Jaeger H.M., Liu C.H. and Nagel S.R., Phys. Rev. Lett. 62 (1989) 40-43.
- [12] Granular Matter, an Interdisciplinary Approach, A. Mehta, Ed. (Springer Verlag, New York, 1993).
- [13] Malkin A.Y., Rheol. Acta 34 (1995) 27-39.
- [14] Bashir Y.M. and Goddard J.D., J. Rheol. 35 (1991) 849-885.
- [15] Bardet J.P. and Proubet J., Geotechnique 41 (1991) 599-613.
- [16] Campbell C.S., J. Fluid Mech. 203 (1989) 449-473.
- [17] Campbell C.S. and Gong A., J. Fluid Mech. 164 (1986) 107-125.
- [18] Louge M.Y., Phys. Fluids 6 (1994) 2253-2257.
- [19] Lun C.K.K. and Bent A.A., J. Fluid Mech. 258 (1994) 335-353.
- [20] Thornton J.D., Acta Mech. 64 (1991) 45-61.
- [21] Walton O.R., Mech. Mater. 16 (1993) 239-247.

- [22] Christoffersen J., Mehrabadi M.M. and Nemat-Nasser S., J. Appl. Mech. 48 (1981) 339-344.
- [23] Cambou C., From global to local variables in granular materials, Powders & Grains, Birmingham 1993, Thornton Eds. (Balkema, Rotterdam, 1993) pp. 73-92.
- [24] Goldshstein A. and Shapiro M., J. Fluid Mech. 282 (1995) 75-114.
- [25] Lun C.K.K., J. Fluid Mech. 233 (1991) 539-559.
- [26] Savage S.B. and Jeffrey D.J., J. Fluid Mech. 110 (1981) 255-272.
- [27] Haff P.K., J. Rheol. **30** (1986) 931-948.
- [28] Goodman M.A. and Cowin S.C., Arch. Rational Mech. Anal. 44 (1971) 249-266.
- [29] Ocone R. and Astarita G., J. Rheol. 37 (1993) 727-742.
- [30] Bedford A. and Dumheller D.S., Int. J. Eng. Sci. 21 (1983) 863-960.
- [31] Berker A. and VanArsdale W.E., Rheol. Acta **31** (1992) 119-138.
- [32] Chapman S. and Cowling T.G., The mathematical theory of non-uniform gases (Cambridge University Press, Cambridge, 1970).
- [33] Lun C.K.K., Savage S.B., Jeffrey D.J. and Chepurnity N., J. Fluid Mech. 140 (1984) 223-256.
- [34] Pöschel T. and Buchholtz V., Phys. Rev. Lett. 71 (1993) 3963-3966.
- [35] Jan C.D., Shen H.W., Ling C.H. and Chen C.I., A sphere moving down an inclined bumpy surface, Proceedings of the Ninth Conf. of Eng. Mech., ISBN 08726228671 (American Society of Civil Engineers,, 1992) pp. 768-771.
- [36] Ristow G.H., Riguidel F.-X. and Bideau D., J. Phys. I France 4 (1994) 1161-1172.
- [37] Ancey C., Coussot P. and Evesque, submitted to Mechanics of Frictional-Cohesive Materials.
- [38] Johnson K.L., Contact Mechanics (Cambridge University Press, Cambridge, 1985) pp. 452.
- [39] Chang W.R. and Ling F.F., J. Tribol. ASME 114 (1992) 439-447.
- [40] Mindlin R.D. and Deresiewicz H., J. Appl. Mech. 21 (1953) 327-344.
- [41] Mindlin R.D., J. Appl. Mech. 71 (1949) 259-268.
- [42] Brach R.M., J. Appl. Mech. 51 (1984) 164-170.
- [43] Brach R.M., J. Appl. Mech. 56 (1989) 133-138.
- [44] Hogue C. and Newland D., Powder Tech. 78 (1994) 51-66.
- [45] Stronge W.J., Proc. R. Soc. Lond. A 421 (1990) 169-181.
- [46] Wang U. and Mason M.T., J. Appl. Mech. 59 (1992) 635-642.
- [47] Lim C.T. and Stronge W.J., Int. J. Mech. Sci. 36 (1994) 911-930.
- [48] Maw N., Barber J.R. and Fawcett J.N., J. Lub. Tech. 103 (1981) 74-80.
- [49] Maw N., Barber J.R. and Fawcett J.N., Wear 1 (1976) 101-114.
- [50] Jaeger J., Appl. Mech. Rev. 47 (1994) 35-54.
- [51] Foerster S., Louge M., Chang H. and Allia K., Phys. Fluids 6 (1994) 1108-1115.
- [52] Smart R., Beimfohr S. and Leighton D.T., Phys. Fluids A 5 (1993) 13-25.
- [53] Byrd P.F. and Friedman M.D., Handbook of elliptic integrals for engineers and physicists (Springer Verlag, Berlin, 1954) pp. 352.
- [54] Martins J.A.C., Monteiro Marques M.D.P. and Gastaldi F., Eur. J. Mech. A/Solids 13 (1994) 113-133.
- [55] Jaeger H.M., Liu C., Nagel S.R. and Witten T.A., Europhys. Lett. 7 (1990) 619-624.
- [56] "Les géomatériaux, Théories, Expériences et Modèles", Darve F., Hicher P.-Y. and Reynouard J.-M., Eds. (Hermès, Paris, 1995) pp. 44-45.
- [57] Campbell C.S., Ann. Rev. Fluid Mech. 22 (1990) 57-92.
- [58] Barker G.C., Computer simulations of granular materials, Granular Matter, an interdisciplinary approach, A. Mehta Eds. (Springer, New York, 1993).