

**1 Stochastic modelling in sediment dynamics: the**  
**2 Exner equation for planar-bed**  
**3 incipient-bedload-transport conditions**

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6 **Abstract.** Even under flow-equilibrium conditions, river-bed topography  
7 continuously evolves with time, producing trains of irregular bedforms. The  
8 idea has recently emerged that the variability in the bedform geometry re-  
9 sults from some randomness in sediment flux. In this paper, we address this  
10 issue by using the Exner equation and a population-exchange model derived  
11 in an earlier paper [*J. Fluid Mech.* **595**, 83–114, 2008]. In this model, par-  
12 ticle entrainment and deposition are idealized as population exchanges be-  
13 tween the stream and the bed, which makes it possible to use birth-death  
14 Markov-process theory to track the number of moving grains. The paper fo-  
15 cuses on nascent bedforms on initially planar beds, a situation in which the  
16 coupling between the stream and bed is weak. In a steady state, the num-  
17 ber of moving particles follows a negative binomial distribution. Although  
18 this probability distribution does not enter the family of heavy-tailed dis-  
19 tributions, it may give rise to large and frequent fluctuations because the stan-  
20 dard deviation can be much larger than the mean, a feature that is not ac-  
21 counted for with classic probability laws (e.g., Hamamori’s law) used so far  
22 for describing bedload fluctuations. In the large-system limit, the master equa-  
23 tion of the birth-death Markov process can be transformed into a Fokker-  
24 Planck equation. This transformation is used here to show that the number  
25 of moving particles can be described as an Ornstein-Uhlenbeck process. An  
26 important consequence is that on long term, the number of moving parti-  
27 cles follows a Gaussian distribution. Laboratory experiments show that this  
28 approximation is correct when the mean number per unit length of stream

29  $\bar{N}/L$  is sufficiently large (typically 2 particles/cm in our experiments). The  
30 particle-number fluctuations give rise to bed-elevation fluctuations, whose  
31 spectrum falls off like  $\omega^{-2}$  in the high-frequency regime (with  $\omega$  the angu-  
32 lar frequency) and variance grows linearly with time. These features are in  
33 agreement with recent observations on bedform development (in particular,  
34 ripple growth).

## 1. Introduction

35 The Exner equation has attracted considerable attention in recent years. This equation  
36 expresses the mass balance of the bed by stating that the local rate of change in the  
37 bed elevation is related to the spatial gradient of the solid discharge [see (14) below].  
38 The equation was termed in honor of Felix Exner, an Austrian physicist, who notably  
39 investigated the development of dunes in sand-bed rivers. In his seminal paper, *Exner*  
40 [1925] assumed that the erosion/deposition rate is proportional to the streamwise gradient  
41 of the flow velocity. With this assumption, he showed that the mass balance equation for  
42 the bed takes the form of a nonlinear hyperbolic advection equation, which accounts for  
43 formation and migration of bedforms [see also *Kubatko and Westerink, 2007*].

44 Much of the early work has considered bed load transport as a continuous process,  
45 i.e. a process that can be conveniently described within the framework of continuum  
46 mechanics in terms of relations between mean values. However, field and laboratory ob-  
47 servations have shown that bed load transport exhibits considerable variability in time and  
48 space, in particular at low flow rates [*Ergenzinger, 1988*]. Field surveys have shown that  
49 bedload-transport time series are highly fluctuating signals, which may be intermittent  
50 and comprise pulses that are more or less correlated with the water discharge [*Carey, 1985*;  
51 *Gomez, 1991*; *Bunte and Abt, 2005*; *Kuhnle and Willis, 1998*; *Ancey et al., 2006*; *Singh*  
52 *et al., 2009*; *Radice, 2009*; *Ganti et al., 2009*]. As a result of sediment transport, alluvial  
53 river beds do not remain planar, but evolve toward a wavy morphology, which is contin-  
54 uously changing with time and gives rise to sediment waves that travel downstream or  
55 upstream depending on the Froude number [*Julien, 1994*; *Coleman and Melville, 1996*].

56 In most investigations of bed dynamics, bedforms are seen as the impact of turbulent  
57 structures on an erodible bed [*Yalin*, 1992] or the consequence of a loss of stability in the  
58 coupled fluid–solid system [*Furbish et al.*, 1998; *Balmforth and Provenzale*, 2001]. More  
59 recently has emerged the idea that bedforms arise from the random nature of particle  
60 trajectories. For instance, using a discrete element model with simple rules for particle  
61 entrainment and motion, *Niño et al.* [2002] simulated individual particle trajectories and  
62 tracked the resulting bed evolution; they found that their model successfully mimicked  
63 many of the physical features of bedforms.

64 These observations of the stochastic nature of bedload transport have given impetus to  
65 a probabilistic formulation of bedload transport equations. While *Einstein* [1936, 1950]  
66 extensively studied some stochastic features of bedload transport, in particular the prob-  
67 ability distribution of heap lengths and travel times, he did not devise a complete prob-  
68 abilistic framework for sediment transport. Today there is renewed interest in working  
69 out an appropriate probabilistic framework, in particular driven by the need to overcome  
70 some limitations of the active-layer concept that underpins most morphodynamic mod-  
71 els. A first step toward this objective was achieved by *Parker et al.* [2000], who used  
72 heuristic arguments to derive a probabilistic formulation of the Exner equation in terms  
73 of bed-elevation-specific entrainment and deposition rates. *Blom et al.* [2008, 2006] and  
74 *Blom and Parker* [2004] elaborated on this model, with emphasis given to river beds cov-  
75 ered by regular spatial patterns (triangular dunes). *Jerolmack and Mohrig* [2005] used a  
76 Langevin-like formulation of the Exner equation: they numerically solved a coupled set of  
77 equations, including the Exner equation where a white-noise term was added to account  
78 for bedload fluctuations. They observed that adding noise in the Exner equation caused

79 the formation of spatial patterns along the bed, which looked like natural bedforms in  
80 that they exhibited a continuous range of sizes and shapes (from ripples to dunes) and  
81 that they grew and moved in a similar way to what it is commonly observed in the field.  
82 *Nakagawa and Tsujimoto* [1980, 1984] showed that entrainment and deposition rates are  
83 in fact related to each other through the probability distribution of step lengths, which  
84 enables consideration of stochastic fluctuations in the Exner equation.

85 Sediment transport is not the only physical system that is driven (to a varying degree)  
86 by fluctuations. Brownian motion, population dynamics, crystal growth, chemical reac-  
87 tions, landscape, earthquakes provide typical examples of stochastic systems [*Turcotte*,  
88 1995; *Sornette*, 2000]. For these extensively studied problems, a longstanding issue has  
89 been to devise computational strategies to model complex systems by distinguishing de-  
90 terministic and stochastic components of the behavior and applying appropriate generic  
91 procedures. There are some analogies that we can draw between sediment transport in wa-  
92 terways and systems made up of a large number of chemically reacting molecules: motion,  
93 entrainment, and deposition can be viewed as reactions between two species (moving and  
94 resting particles), as shown in Appendix B. In so doing, we can use all the mathematical  
95 machinery developed to build up a consistent theoretical framework and work out com-  
96 putational tools (Fokker-Plank equation, approximation by Ornstein-Uhlenbeck process,  
97 large-system limit expansion) [*Gardiner*, 1983]. A lesson learned in chemical kinetics is of  
98 particular relevance to our issue: in a well-stirred and large system (i.e., a system made up  
99 of a large number of molecules occupying all the available space), how the concentration  
100 of each chemical species evolves in time can be described using deterministic, first-order  
101 ordinary differential equations (called the reaction-rate equations) [*Gillespie*, 2007]. This

102 approximation breaks when molecules are present in small numbers, when reactions occur  
103 at low rates, or when molecules are heterogeneously distributed in the system; in that case,  
104 large fluctuations affect the system evolution to the point that dynamics are markedly  
105 different from the predictions of reaction-rate equations.

106 The objective of this paper is to recast the Exner equation in a probabilistic form  
107 (Langevin-like equation). Given the complexity of this issue, we will focus here on planar  
108 beds along which bedforms (nascent waves, ripples) start to develop. To that end, we  
109 will use an Eulerian approach and idealize particle entrainment and deposition as a birth-  
110 death Markov process, which make it possible to track the time variation in the number  
111 of moving particles  $N(t)$  inside a fixed control volume  $\mathcal{V}$ . The statistical properties of the  
112 time evolution of  $N$  can be investigated using Markov process theory; this was presented  
113 at length in an earlier paper [*Ancey et al.*, 2008]. Here, we are interested in determining  
114 how these properties change in the large-system limit (i.e., when the size of  $\mathcal{V}$  is increased).  
115 The consequences in terms of sediment mass balance can then be inferred.

116 To begin with, we outline the statistical model developed in [*Ancey et al.*, 2008] and  
117 compare it with other stochastic approaches, most of them being Lagrangian descriptions  
118 of particles experiencing random walks. A key issue is the definition of the solid discharge.  
119 As shown in Appendix A, there are many equations that express sediment flux, but there  
120 is no guarantee that they are all compatible and can equally serve within a probabilistic  
121 framework. We use a relation that links the solid discharge with the number of moving  
122 particles. A generalized birth-death Markov process is used to compute the number of  
123 moving particles  $N(t)$  as well as the entrainment and deposition rates; see also [*Ancey*  
124 *et al.*, 2008] and Appendix B for details. In § 3.2, we will see how the stream bed evolves

125 under our model. We focus our attention to initially planar beds and low sediment  
126 transport rates (i.e. for bottom shear stresses slightly in excess of the threshold of incipient  
127 motion); as a result of sediment transport, bedforms start to appear in the form of ripples  
128 (i.e., bedforms with wave heights less than a few centimeters) [*Coleman and Nikora,*  
129 2009]. Bed dynamics is accounted for through the Exner equation. When the system  
130 contains a sufficiently large number of particles, the master equation, which specifies  
131 how the probability of  $N(t)$  evolves with time, can be conveniently transformed into  
132 a Fokker-Planck equation, which considers  $N(t)$  as a continuous random variable rather  
133 than a discrete one. This transformation makes it possible to derive a probabilistic variant  
134 of the Exner equation in §3.2. The discrete and continuous probability distributions  
135 of  $N(t)$  are then compared with experiments. In the few experiments presented here,  
136 flow conditions are close to those encountered with shallow mountain streams and coarse  
137 sediment, but in principle our framework holds for a wide range of waterways provided  
138 that the physical assumptions used in the model (sediment flux is due to bedload transport  
139 and not suspended load) make sense. Except for some crude analytical approximations,  
140 there is no analytical solution to the stochastic differential equations that specify the  
141 system evolution. Numerical schemes must then be used (see §3.6); I here provide proof  
142 of concept without dwelling on the algorithmic aspects.

143 The notation used in this paper tries to be as much standard as possible. Greek letters  
144 generally denote parameters, e.g., entrainment and deposition coefficients in our model.  
145 Roman letters are used for random variables and the particular value they can take is  
146 denoted by lowercase letters. For instance,  $N$  is a random variable describing the number  
147 of moving particles; the probability that  $N$  takes the value  $n$  is denoted by  $P(n) =$



148  $\Pr(N = n)$ . The average or expectation of a distribution is denoted by a bar, e.g.,  $\bar{N}$  is  
149 the mean number of particles; for the sake of simplicity, I use the same symbol to refer to  
150 sample averages and theoretical expectations. The variance is denoted  $\text{var}$ . The symbol  
151  $o$  is a shorthand notation to for ‘much smaller than’, e.g.  $u = o(v)$  means that  $u \ll v$ .  
152 The symbol  $O$  indicates that there is a one-sided bound, e.g.,  $u = O(v)$  means that the  
153 limit of  $u/v$  exists and is finite (neither zero nor infinity). Here probability functions  $P$   
154 are probability density functions (or probability mass functions for discrete variables). A  
155 variable index can be found at the end of this paper.

## 2. Model outline

### 2.1. Earlier results

156 We have recently developed a model that computes the number of moving particles  
157 inside a control volume [Ancey *et al.*, 2008]. The problem we addressed was an idealization  
158 of bedload transport in a mountain stream, where sediment was replaced by spherical  
159 particles of equal size and the problem was purely two-dimensional. We adopted an  
160 Eulerian point of view by considering a fixed volume of control. Figure 1 depicts the  
161 control volume in which we count the number  $N$  of particles. This number  $N$  varies  
162 with time as a result of inflow, outflow, deposition, and entrainment of particles from  
163 the bed. More specifically, we assumed that particles enter and leave the window at  
164 rates  $\nu_{in}$  and  $\nu_{out}$  (in beads/s and 1/s, respectively). In other words, the probability that  
165 one particle enters the window within a small time increment  $\delta t$  is  $\nu_{in}\delta t$ ; the probability  
166 that two particles enter the window at the same time or over short time spans  $\delta t$  is  
167 zero, which is a reasonable assumption for dilute to moderately concentrated bedload  
168 transport. For outflow, the probability that a given bead leaves the window is  $\nu_{out}\delta t$ ,

169 but since there are  $N = n$  particles within the window, the probability that any particle  
 170 leaves the window is  $n\nu_{out}\delta t$ . For entrainment, we consider two processes: a particle can  
 171 be dislodged from the bed as a result of fluid action (this is the classical assumption  
 172 in sediment transport), but it can also be destabilized because of the moving particles,  
 173 which can interact directly (collision) or indirectly (wake effect, advection of turbulent  
 174 structure) with the bed particles. Individual entrainment occurs at rate  $\lambda$ , while collective  
 175 entrainment is at rate  $\mu$ ; entrainment rate is then dependent on the number of moving  
 176 particles:  $E \propto \lambda + \mu N$ . For deposition, we could also distinguish various processes, but  
 177 they are all dependent on  $N$  and therefore, we just refer to  $\sigma$  as the total deposition rate.

178 Particle exchanges can be described using a birth-death emigration-immigration Markov  
 179 model (see Appendix B). Under steady-state conditions, several interesting features can  
 180 be derived. First of all, when  $\mu > 0$  it can be shown that the probability of finding  $N = n$   
 181 particles in the control volume is the negative binomial distribution

$$P(n) = \text{NegBin}(n; r, p) = \frac{\Gamma(r+n)}{\Gamma(r)n!} p^r (1-p)^n, n = 0, 1, \dots, \quad (1)$$

182 with  $r = (\lambda + \nu_{in})/\mu$  and  $p = 1 - \mu/(\sigma + \nu_{out})$ , and where  $\Gamma$  denotes the gamma function.

183 The mean and the variance are given, respectively, by

$$\bar{N} = r \frac{1-p}{p} = (\lambda + \nu_{in})/(\sigma + \nu_{out} - \mu) \quad \text{and} \quad \text{var}N = r \frac{1-p}{p^2} = \frac{(\lambda + \nu_{in})(\sigma + \nu_{out})}{(\sigma + \nu_{out} - \mu)^2}. \quad (2)$$

184 When  $\mu = 0$ ,  $P(n)$  is the Poisson distribution and in that case, the variance equals the  
 185 mean.

186 Although the negative binomial distribution does not belong to the family of heavy-tail  
 187 distributions (i.e., with an infinite variance), it has a tail falling off like  $n^r(1-p)^n$ . Its  
 188 decay rate is thus slower than for an exponential distribution. Furthermore, when  $p \rightarrow 0$ ,

189 the coefficient of variation  $\sqrt{\text{var}/\bar{N}}$  tends to  $\sqrt{r}$ . This coefficient of variation can be much  
 190 larger than unity, giving rise to large fluctuations and the appearance of a heavy-tail be-  
 191 havior. Note that this behavior is not shared by other classical discrete distributions such  
 192 as the Poisson or binomial distribution. Continuous probability distributions used specifi-  
 193 cally in sediment transport for describing fluctuations such as Hamamori's distribution (5)  
 194 are also characterized by a constant coefficient of variation (see next subsection). Here,  
 195 the potentially large variance results from what we referred to as *collective* entrainment.  
 196 If we set  $\mu$  to 0, we retrieve an Einstein-like behavior, with a Poissonian distribution for  
 197  $N$  [Ancey *et al.*, 2006, 2008]. Note also that in our model, the jump probability depends  
 198 on the number of moving particles (inside the control volume), which is affected by im-  
 199 migration and emigration, i.e. by its neighborhood. It thus describes a non-local process,  
 200 where the local state depends on inflow and outflow.

201 The second key feature is the autocorrelation function  $R(t)$  of the number of particles  
 202 in motion (within the control volume), which is exponential

$$R(t) = e^{-t/t_c}, \quad (3)$$

203 with  $t_c = 1/(\sigma + \nu_{out} - \mu)$  the autocorrelation time. When  $\mu > 0$ , the autocorrelation  
 204 time is quite long compared to the characteristic time of a moving particle  $t_m = 1/\sigma$ ,  
 205 which probably reflects a memory effect due to the coupling between entrainment and  
 206 the number of moving particles. This also shows that even though birth-death Markov  
 207 models are based on vanishing time increments  $\delta t$  in the counting process, the typical  
 208 timescale of  $N$  is very large compared to particle timescale ( $t_m$  or  $\delta t$ ).

209 The third feature that can be derived under this model is the waiting time between  
 210 events. Once in motion, a particle continues moving for a random, exponentially distrib-

211 uted time (with parameter  $\sigma$ ); the time for which the particle is stationary has a far more  
 212 complicated distribution, which cannot be specified in practice here. Indeed, there are  
 213 major impediments to this determination: in our Eulerian approach, the details of a par-  
 214 ticle's history are ignored; furthermore, the waiting time for entrainment depends on the  
 215 depth at which the particle was buried in the bed. Therefore, contrary to diffusion theory  
 216 (see below), there is no clear relation between Lagrangian and Eulerian points of view  
 217 within our framework. Other waiting times can be more easily computed. For instance,  
 218 the waiting time between two deposition events within the control volume is distributed  
 219 exponentially with a rate  $1/(n\sigma)$ .

220 There is no unique way to define the solid discharge and the statistical properties of the  
 221 solid discharge depend greatly on the form selected. In Appendix A, we review the main  
 222 forms that are currently of common use and comment on their strengths and weaknesses.  
 223 Here we define the sediment transport rate  $\dot{n} = Q_s/v_p$  as

$$\dot{n} = \frac{1}{L} \sum_{i=1}^N u_i = \frac{N}{L} \bar{u}_p, \quad (4)$$

224 where  $\bar{u}_p$  denotes the mean particle velocity,  $v_p$  is the particle volume, and  $Q_s$  is the total  
 225 solid discharge. Fluctuations in the solid discharge arise from the variations in the number  
 226 of moving particles in the control volume as well as their velocities. Surprisingly enough,  
 227 we observe that the particle velocity  $\bar{u}_p$  weakly contributes to the solid-discharge variance  
 228 when the bed is mobile [*Böhm et al.*, 2004; *Ancey et al.*, 2008]; furthermore, contrary  
 229 to the case in which a single particle is moving, collective motion of particles implies a  
 230 weak dependence of particle velocity on bottom shear stress (see Appendix A). For a  
 231 slope range of 7.5%–15% and a sufficiently wide range of bottom shear stress (above the  
 232 threshold of incipient motion), *Ancey et al.* [2008] observed no dependence of the particle

233 velocity for the rolling regime and weak dependence for the saltating regime; moreover the  
 234 statistical properties of  $N$  and  $Q_s$  (e.g., the shape of the empirical probability distribution  
 235 function and the autocorrelation function) were very close, confirming that  $N$  is the key  
 236 to understanding bedload fluctuations at low sediment-transport rates. Using a wide  
 237 flume, Schmeeckle obtained the same results concerning the weak dependence of  $\bar{u}_p$  on  
 238  $\tau_b$  (M.W. Schmeeckle, personal communication 2007). These experimental observations  
 239 prompt the use of  $N$  as the main factor explaining the  $Q_s$  variance. More complicated  
 240 dependences can be accounted for without significantly altering the structure of the model  
 241 here. For instance, *Böhm et al.* [2004] showed how to combine both  $N$  and  $\bar{u}_p$  fluctuations  
 242 to compute the statistical properties of  $\dot{n}$ .

## 2.2. Comparison with other approaches

243 There are numerous approaches to computing the solid discharge, but only a few yield  
 244 both the dependence on water flow rate and its statistical properties. *Einstein* [1936, 1950]  
 245 devised a simple model where bedload transport results from the imbalance between en-  
 246 trainment and deposition rates. Fluctuations arise from the intermittent motion of parti-  
 247 cles: once moving, particles perform steps of random length and once they are deposited,  
 248 the periods of rest are also of random span. Note that setting  $\mu = 0$  (i.e. no collective  
 249 entrainment) in our model leads to an Einstein-like model. This model has seen many  
 250 refinements, the most popular being Hamamori's distribution which gives the probability  
 251 density function of the scaled variables  $z = (Q_s - \bar{Q}_s)/\sqrt{\text{var } Q_s}$

$$P(z) = -\frac{\sqrt{7}}{12} \ln \left( \frac{\sqrt{7}}{12} z + \frac{1}{4} \right), \quad (5)$$

252 where  $\bar{Q}_s$  is the averaged solid discharge and the variance is related to this mean value  
 253 by:  $\text{var } Q_s = 7\bar{Q}_s^2/9$ , that is, a coefficient of variation of  $\sqrt{7}/3$  [Carey, 1985; Kuhnle and  
 254 Willis, 1998]. Note that this distribution is bounded since  $-3/\sqrt{7} < z < 9/\sqrt{7}$ , which  
 255 imposes that the maximum amplitude of fluctuations (around the mean value) is four  
 256 times the mean rate.

257 In recent years, growing attention has been paid to models based on diffusion theory.  
 258 Originally, linear diffusion theory was used to model erosion processes in sedimentary  
 259 basins over geological timescales [e.g., see Paola *et al.*, 1992]. In the particular case of  
 260 a one-dimensional stream, the governing equation takes the form of a parabolic partial  
 261 differential equation

$$\frac{\partial b(x, t)}{\partial t} = \frac{\partial}{\partial x} \left( K \frac{\partial b(x, t)}{\partial x} \right), \quad (6)$$

262 with  $b(x, t)$  the bed elevation and  $K$  is a constant reflecting soil diffusivity. Field surveys  
 263 giving little hope for linearity [Heimsath *et al.*, 2005], emphasis was then given to nonlinear  
 264 forms of (6), where  $K$  is assumed to be a function of  $b$  [Postma *et al.*, 2008], and anomalous  
 265 diffusion, where derivatives are replaced with fractional derivatives [Voller and Paola,  
 266 2009; Stark *et al.*, 2009].

267 It is also very tempting to use advection-diffusion equations to model sediment transport  
 268 on shorter timescales than geological times [Graf, 1984]. For instance, the following  
 269 equation describes how the concentration in particles varies with time and position

$$\frac{\partial c}{\partial t} = -u \frac{\partial c}{\partial x} + \frac{\partial}{\partial x} \left( K \frac{\partial c}{\partial x} \right), \quad (7)$$

270 where  $u$  is the mean velocity at which the particles are transported and  $K$  is a (constant)  
 271 diffusion coefficient. A particular strength of this formulation is that it can be derived

272 using an Eulerian continuum-based approach or a Lagrangian discrete-particle approach.  
 273 The correspondence between both approaches is detailed at length by *Schumer et al.*  
 274 [2009], but it is worth recalling some fundamentals here. The solid concentration is the  
 275 volume fraction occupied by particles and is related to the probability  $P_1(x, t)$  of finding  
 276 a particle placed at  $x$  at time  $t$  [*Herczynski and Pienkowska, 1980*]:

$$c(x, t) = NP_1(x, t), \quad (8)$$

277 with  $N$  the number of particles. Solving (7) for  $c$  is then equivalent to finding the prob-  
 278 ability  $P_1(x, t)$ , which gives an unusually simple relation for connecting probabilistic and  
 279 deterministic descriptions. If the flow of particles is dilute, we can reasonably assume that  
 280 the motion of a single particle can be closely described using a random walk: the particle  
 281 randomly jumps from one position to another one. The jumps of length  $X_i$  are randomly  
 282 distributed with mean  $m$  and finite variance  $v$ . There is no waiting time between jumps.  
 283 The times of flight are constant and equal to  $\Delta t$ . The distance  $d$  travelled by the particle  
 284 at time  $t$  is

$$d(t) = \sum_{i=1}^k X_i \quad (9)$$

285 where  $k$  is the number of jumps achieved until time  $t$ ; for sufficiently large values of  $t$ ,  
 286 we have  $k \approx t/\Delta t$ . The law of large numbers states that the sum of independent and  
 287 identically distributed random variables (with finite variance  $v$ ) scaled by  $\sqrt{vk}$  converges  
 288 to the standard normal distribution  $\mathcal{N}(0, 1)$  (of mean 0 and variance 1)

$$\frac{\sum_{i=1}^k X_i - km}{\sqrt{vk}} \rightarrow \mathcal{N}(0, 1), \quad (10)$$

289 which can be recast in the following form

$$d(t) \approx \frac{m}{\Delta t}t + \sqrt{\frac{v}{\Delta t}}\sqrt{t}\mathcal{N}(0, 1). \quad (11)$$

290 This shows that the probability  $P_1(x, t)$  of finding the particle at  $x = d$  is given by a normal  
 291 law with mean  $m/\Delta t$  and variance  $v/\Delta t$ . The random-walk model can be made more  
 292 complex, e.g., by considering infinite variance for particle jumps or by allowing random  
 293 waiting times between jumps. This results in fractional advection-diffusion equations,  
 294 widely used to investigate solute transport in groundwater [Berkowitz *et al.*, 2006]. A key  
 295 feature is that the probability distribution  $P_1$  (or the solid concentration) is a heavy-tailed  
 296 distribution, which explains why wide and frequent fluctuations occur in solute transport.  
 297 For instance, if the distribution of particle step lengths is heavy-tailed (i.e. with infinite  
 298 variance), then  $P_1$  or  $c$  satisfies a fractional-in-space advection-diffusion equation, which  
 299 generalizes the classic advection-diffusion equation [Metzler and Klafter, 2004; Schumer  
 300 *et al.*, 2009]:

$$\frac{\partial c}{\partial t} = u \frac{\partial c}{\partial x} + K \frac{\partial^\gamma c}{\partial x^\gamma}, \quad (12)$$

301 where  $0 < \gamma < 2$  is the fractional exponent and we introduced the fractional derivative

$$\frac{\partial^\gamma c}{\partial x^\gamma} = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{\partial c(x-y, t)}{\partial x} y^{-\beta} dy. \quad (13)$$

302 The solution to (12) has no analytical expression, but one can infer its main features: it  
 303 is positively skewed and has a tail that falls off like  $x^{-\gamma-1}$ , i.e. much slower than the tails  
 304 of exponential or Gaussian distributions. Note that since the fractional derivative (13) is  
 305 equivalent to a convolution with a power-law weighting function, fractional diffusion is a  
 306 nonlocal process [Tucker and Bradley, 2009].

307 The question arises as to whether the fractional-diffusion framework will help improve  
 308 our understanding of bedload transport. Although there is abundant literature in porous-  
 309 medium and groundwater problems on fractional diffusion equations and random walks,



310 there are not many results available for sediment transport. Much of the literature has  
311 focused on tracers in gravel-bed rivers and flumes. By observing the final distribution  
312 of particle displacements, *Einstein* [1936] and *Hubbell and Sayre* [1964] deduced that the  
313 deposition patterns are consistent with a gamma distribution of step lengths. On the  
314 whole, subsequent field surveys led to similar conclusions, but they also pointed out that  
315 data were noisy and the final distribution of particle displacements could be complicated  
316 since particles could be trapped in river bars and their movement could be affected by  
317 local hydraulic conditions depending on channel morphology [*Hassan and Church*, 1991;  
318 *Ergenziger and Schmidt*, 1992; *Pryce and Ashmore*, 2003; *Ferguson et al.*, 2002; *McNamara*  
319 *and Borden*, 2004]. More recently, new techniques such as radio tracking has made it  
320 possible to follow up the travel path of individual particles; *Habersack* [2001] reported that  
321 for observation timescales of a few minutes, the rest periods are exponentially distributed,  
322 while the distribution of step lengths is closely approximated by a gamma distribution.  
323 In contrast with other authors, *Nikora et al.* [2002] focused on particle movement as a  
324 function of time rather than step-length distribution at a given time. From their flume  
325 experiments, they deduced that  $\text{var } d(t) \propto t^\gamma$  with  $\gamma \geq 2$  for short timescales (typically  
326 on the order of a few seconds), but  $\gamma < 2$  for long timescales (typically on the order  
327 of a few minutes), which means that particle displacements exhibit anomalous diffusive  
328 properties, either superdiffusive or subdiffusive depending on the timescale. This behavior  
329 can be understood by considering that the longer the timescale is, the more likely the  
330 particle is to be trapped in the bed. In summary, while there is growing evidence of wide  
331 and frequent fluctuations in bedload transport (in particular at low flow rates) [*Carey*,  
332 1985; *Kuhnle and Southard*, 1988; *Kuhnle and Willis*, 1998; *Bunte and Abt*, 2005; *Ancey*

333 *et al.*, 2006; *Singh et al.*, 2009; *Radice*, 2009], the origins of these fluctuations are not  
334 clear cut: while there are some data supporting the idea of fractional diffusion [*Nikora*  
335 *et al.*, 2002], the few data collected so far in the field or laboratory do not reveal typical  
336 fractional-diffusion behavior for particle displacements or bedload concentration.

### 3. Bed dynamics and Exner equation

#### 3.1. Setting

337 We consider a two-dimensional, steady water stream flowing down an erodible bed  
 338  $y = b(x, t)$ , with  $x$  the downstream spatial coordinate,  $t$  time, and  $y$  the vertical coordinate  
 339 measured from a fixed reference level. The  $x$ -axis is tilted at a constant angle  $\theta$  to the  
 340 horizontal. Moving and resting particles are spherical particles of equal radius  $a$  and  
 341 density  $\rho_p$ . The free surface is at elevation  $y = s(x, t)$  and the flow depth  $h(x, t) =$   
 342  $s(x, t) - b(x, t)$  is measured normal to the reference level (i.e. in the  $y$ -direction). Figure 2  
 343 shows the flow configuration.

#### 3.2. Exner equation and bed dynamics

344 Even under bed-equilibrium conditions, the bed does not remain flat and bedforms  
 345 develop. We must then use an equation specifying the bed mass balance. When the  
 346 suspended load stored in the water column is negligible compared to bed load, the Exner  
 347 equation fulfills this objective

$$(1 - \phi_p) \frac{\partial b}{\partial t} = D - E = -\frac{\partial q_s}{\partial x}, \quad (14)$$

348 where  $\phi_p$  is the bed porosity,  $D$  ( $E$ , respectively) denotes the volume rate of deposition  
 349 (entrainment, respectively) per unit time and per unit bed surface onto the bed, and  $q_s$   
 350 represents the solid discharge per unit width.

351 The entrainment and deposition rates are assumed to be single-valued functions of the  
 352 bottom shear stress:  $D = D(\tau_b)$  and  $E = E(\tau_b)$ . Under steady uniform flow conditions,  
 353 the bottom shear stress is  $\tau_b(h) = \rho g h \sin \theta$ . Here we focus on planar-bed lower-regime flow  
 354 conditions, which describe situations where the bed remains nearly planar even though  
 355 bedforms start to appear as a result of incipient bedload transport [*Julien, 1994; Coleman*

356 *and Nikora, 2009*]. The water stream is in a nearly uniform or gradually varied regime and  
 357 is not yet disturbed by nascent bedforms. Sediment transport occurs at low rate, i.e. the  
 358 bottom shear stress  $\tau_b$  just exceeds the threshold of incipient motion  $\tau_c$  or in other words,  
 359 the transport-stage parameter  $T$  is low, typically  $T < 3$  with  $T = \tau_b/\tau_c - 1$  [*van Rijn,*  
 360 1984]. In this case, the bottom shear stress remains, as a first approximation, very close  
 361 to its value in a steady uniform regime. A first-order correction of the uniform-regime is

$$\tau_b(\bar{u}, b, h) = \rho g h \cos \theta \left( \tan \theta - \frac{\partial b}{\partial x} - \frac{\partial h}{\partial x} \right), \quad (15)$$

362 which accommodates for both slight changes in bed gradient  $\partial_x b$  and flow-depth deviations  
 363 from the uniform regime  $\partial_x h$  [*Julien, 1994*].

364 Under our model, the entrainment and deposition rates are found to depend on the  
 365 number of moving particles  $N$  as follows:

$$E = \frac{v_p}{L}(\lambda + \mu N), \text{ and } D = \frac{v_p}{L}\sigma N, \quad (16)$$

366 where  $\lambda$  (in beads/s),  $\mu$  (in  $\text{s}^{-1}$ ), and  $\sigma$  (in  $\text{s}^{-1}$ ) are functions of  $\tau_b$ ,  $v_p = 4\pi a^3/3$  denotes  
 367 the particle volume and  $L$  is the length of the control volume. The entrainment rate  $\lambda$   
 368 is also proportional to  $L$ , while the exchange rates  $\sigma$  and  $\mu$  are independent of it. We  
 369 can then introduce an entrainment rate per unit length ( $\tilde{\lambda}$  in beads  $\text{m}^{-1} \text{s}^{-1}$ ), a scaled  
 370 outflow rate ( $\tilde{\nu}_{out}$  in  $\text{m}^{-1}\text{s}^{-1}$ ), and the concentration of moving particles per unit length  
 371 ( $\tilde{N}$  in beads  $\text{m}^{-1}$ ):

$$\lambda = L\tilde{\lambda}, \nu_{out} = \frac{\tilde{\nu}_{out}}{L}, \text{ and } N = L\tilde{N}. \quad (17)$$

372 This scaling holds since (i) for entrainment, the dependence on  $L$  is encoded through  $N$   
 373 for collective entrainment ( $\mu$ ), while for individual entrainment, the longer the window is,  
 374 the more probable particle entrainment is; (ii) for outflow, as illustrated by Fig. 5 in § 3.4

375 or more rigorously by computing intercorrelation times [*Böhm et al.*, 2004], the outflow  
 376 rate is related to the mean particle velocity (i.e.,  $\tilde{\nu}_{out} \approx u_p$ ). We can then express the  
 377 entrainment and deposition rates as follows

$$E = \frac{v_p}{L}(\lambda + \mu N) = v_p(\tilde{\lambda} + \mu\tilde{N}) \text{ and } D = \frac{v_p}{L}\sigma N = v_p\sigma\tilde{N}. \quad (18)$$

378 The Exner equation (14) becomes

$$\psi \frac{\partial b}{\partial t} = (\sigma - \mu)\tilde{N} - \tilde{\lambda}, \quad (19)$$

379 with  $\psi = 1 - \phi_p$ . Equation (19) is a stochastic differential equation because  $\tilde{N}$  is a  
 380 random variable, the variations of which are described by the master equation (20) or  
 381 the approximate governing equation (27), as shown in the next subsection. This makes  
 382 the analysis and numerical treatment of (19) somewhat delicate and prompts us to seek  
 383 analytical approximation of  $N$ .

### 3.3. Behavior of $N$ in the large-system limit

384 The model outlined in §2.1 is a birth-death Markov process, which allows jumps only  
 385 one step up or down [*Cox and Miller*, 1965] over short time spans. For  $n \geq 1$ , the  
 386 governing equation that specifies the probability of finding  $N = n$  particles in the control  
 387 volume is

$$\frac{\partial}{\partial t}P(n; t) = (n + 1)\alpha P(n + 1; t) + \{\beta + (n - 1)\mu\} P(n - 1; t) - \{\beta + n(\alpha + \mu)\} P(n; t), \quad (20)$$

388 with the short-hand notation  $\alpha = \sigma + \nu_{out}$  and  $\beta = \lambda + \nu_{in}$ . For  $n = 0$ , we have

$$\frac{\partial P(0, t)}{\partial t} = \alpha P(1; t) - \beta P(0; t). \quad (21)$$

389 At time  $t = 0$ , there are  $N = N_0$  particles within the control volume, so we set

$$P(n; 0) = \delta(n - N_0), \quad (22)$$

390 where  $\delta$  is the Kronecker delta function. The forward master equation (20) offers some  
 391 resistance to analysis. There are some general techniques based on an appropriate trans-  
 392 formation of the dependent variable (e.g., Laplace transform and generating functions) to  
 393 find analytical solutions to the differential-difference equation (20), but the inverse trans-  
 394 formation is not always easy to handle; for instance in [Ancey *et al.*, 2008], we provided a  
 395 general solution to (20) in terms of the generating function  $G(z, t) = \sum_{n=0}^{\infty} z^n P(n; t)$ . For  
 396 steady flow conditions, it can be shown that the solution to (20) is a negative binomial  
 397 distribution (1).

398 Here we proceed differently by seeking an analytical approximation of (20) for suffi-  
 399 ciently large  $N$  values. The idea is to transform (20) into a Fokker-Plank equation, which  
 400 provides further insight into the stochastic behavior of bedload transport. There are stan-  
 401 dard techniques (Kramers-Moyal and van Kampen expansions) described in the technical  
 402 literature to achieve such a transformation [Gardiner, 1983]. Using a less rigorous, but  
 403 far easier and more intuitive approach, we will proceed differently: essentially, we in-  
 404 terpret the contributions  $P(n + 1)$  and  $P(n - 1)$  as the terms arising in a Taylor series  
 405 expansion of  $P(n)$  since to first order and  $n$  sufficiently large (compared to unity), we  
 406 have  $P(n + 1) - P(n - 1) = 4P'(n)$  and  $P(n + 1) - 2P(n) + P(n - 1) = P''(n)$ . In the  
 407 large-number limit, we find that the forward master equation (20) can be approximated  
 408 by

$$\frac{\partial}{\partial t} P(n; t) = -\frac{\partial}{\partial n} (AP) + \frac{1}{2} \frac{\partial^2}{\partial n^2} (BP), \quad (23)$$

409 where  $A = \beta + n(\mu - \alpha)$  is called the *drift function* and  $B = (\alpha + \mu)n + \beta$  is the *diffusion*  
 410 *function*. Function  $A$  describes the deterministic component, while function  $B$  reflects the  
 411 random fluctuations of sediment transport rate. Indeed when  $B$  is zero, (23) is strictly  
 412 equivalent to the nonlinear deterministic (Liouville) differential equation  $dN/dt = A(N)$   
 413 [see (29) thereafter]; when  $A$  is zero and  $B$  is constant, (23) describes a special Wiener  
 414 process (i.e. the random walk of a Brownian particle). A net advantage of (23) compared  
 415 to (20) is that we better understand how the different exchanges between the bed and the  
 416 stream gives rise to effects that can be interpreted as deterministic process and stochastic  
 417 diffusion at the macroscopic scale [see also Chap. 6, *Gillespie*, 1992, for further discussion].

418 Another equivalent and useful form of the Fokker-Planck equation (23) is the (Ito)  
 419 stochastic differential equation [*Gardiner*, 1983]:

$$dN = A(N)dt + B^{1/2}(N)d\mathcal{W}(t), \quad (24)$$

420 where  $\mathcal{W}(t)$  is the Wiener process. In the following, we will use this equation in the Exner  
 421 equation to evaluate the effect of  $N$ , but prior to this, we will make a change of variables  
 422 that transforms the discrete random variable  $N$  into a continuous one  $\tilde{N}$ . We can express

$$A = L \left( \tilde{\lambda} + \frac{\nu_{in} - \tilde{N}\tilde{\nu}_{out}}{L} + \tilde{N}(\mu - \sigma) \right) \text{ and } B = L \left( (\sigma + \mu)\tilde{N} + \tilde{\lambda} + \frac{\nu_{in} + \tilde{N}\tilde{\nu}_{out}}{L} \right) \quad (25)$$

423 which shows that in the large-system limit ( $L \rightarrow \infty$ ), the following approximations hold  
 424 to first order

$$\tilde{A} = \frac{A}{L} = \tilde{\lambda} + \tilde{N}(\mu - \sigma) + O(1/L) \text{ and } \tilde{B} = \frac{B}{L} = \tilde{\lambda} + (\sigma + \mu)\tilde{N} + O(1/L). \quad (26)$$

425 This change of variables turns out very helpful in that the system size appears in the  
 426 governing equations, which is crucial to simplifying these equations. Equation (24) can

427 now be expressed as a function of  $\tilde{N}$ :

$$d\tilde{N} = \tilde{A}(\tilde{N})dt + \frac{1}{\sqrt{L}}\tilde{B}^{1/2}(\tilde{N})d\mathcal{W}(t). \quad (27)$$

428 In the large-system limit, the factor  $\epsilon = L^{-1/2}$  tends toward zero, implying that the  
 429 stochastic fluctuations generating weak noise in the time evolution of  $\tilde{N}$  cancel out; in  
 430 other words, the system is entirely driven by the deterministic component  $\tilde{A}$ . This leads  
 431 to posing a power-series expansion of  $\tilde{N}$

$$\tilde{N} = \tilde{N}_* + \epsilon\eta_1(t) + \dots \quad (28)$$

432 The zero-order term in this expansion is also called the *sure function*; it is the solution to  
 433 the Liouville equation

$$\frac{d\tilde{N}}{dt} = \tilde{A}(\tilde{N}), \quad (29)$$

434 subject to an initial condition in the form  $\tilde{N} = \tilde{N}_0$  at  $t = 0$ . When the bottom shear  
 435 stress is constant or does not deviate significantly from its mean value, the solution is

$$\tilde{N}_*(t) = \frac{\tilde{\lambda} - e^{-t(\sigma-\mu)}(\tilde{\lambda} + N_0(\mu - \sigma))}{\sigma - \mu}. \quad (30)$$

436 The number of moving particles decreases exponentially toward a constant value. The  
 437 solution  $\tilde{N}_*(t)$  represents the deterministic component of the behavior when the stochastic  
 438 fluctuations are ignored. Note that at long times and for constant bottom shear stress,  
 439 the number of moving particles approaches a stationary value

$$\tilde{N}_*(t) \rightarrow \tilde{N}_\infty = \frac{\tilde{\lambda}}{\sigma - \mu}, \quad (31)$$

440 which is the steady-state mean number [Ancey *et al.*, 2008]. For large systems, the relax-  
 441 ation time is  $t_r = (\sigma - \mu)^{-1}$ .



442 The first-order correction term  $\eta_1$  is the solution to

$$d\eta = (\mu - \sigma)\eta(t)dt + \tilde{B}^{1/2}(N_*(t))d\mathcal{W}(t), \quad (32)$$

443 subject to an initial condition in the form  $\eta = 0$  at  $t = 0$ . Note that the diffusion function  
 444 is now  $B^{1/2}(N_*(t))$ , which implies that the fluctuation amplitude is imposed by the sure  
 445 function  $N_*$ . From (32), we deduce that  $\eta_1$  is a continuous Markov process with drift  
 446 function  $(\mu - \sigma)\eta_1(t)$  and diffusion function  $\tilde{B}(N_*(t))$ ; it represents additive noise. Since  
 447 the drift function is linear and the diffusion function is constant when  $t \rightarrow \infty$ , we deduce  
 448 that the long-time behavior of  $\eta_1$  can be approximated by an Ornstein-Uhlenbeck process  
 449 with parameter  $k_o = \sigma - \mu$  and diffusion  $D_o = \tilde{B}(\tilde{N}_\infty)$  [*Gardiner*, 1983, see chap. 3].  
 450 Since the long-time behavior of a Ornstein-Uhlenbeck process is a Gaussian process  $\mathcal{N}$   
 451 with mean 0 and variance  $D_o/(2k_o)$ , we deduce that

$$t \rightarrow \infty, \eta_1 \sim \mathcal{N}\left(0, \frac{D_o}{2k_o}\right) = \mathcal{N}\left(0, \frac{\tilde{\lambda}\sigma}{(\sigma - \mu)^2}\right). \quad (33)$$

452 In other words, this also means that in the limit of  $t \rightarrow \infty$ , the number of moving particles  
 453 follows a Gaussian distribution

$$\tilde{N} \sim \mathcal{N}\left(\tilde{N}_\infty, \epsilon^2 \frac{\tilde{\lambda}\sigma}{(\sigma - \mu)^2}\right), \quad (34)$$

454 or equivalently  $n \sim \mathcal{N}\left(N_\infty, \frac{\lambda\sigma}{(\sigma - \mu)^2}\right)$ . The standard deviation of the particle concentration  
 455  $\tilde{N}$  goes to zero with increasing  $L$  like  $L^{-1/2}$ ; this is the usual dependence on system size  
 456 for thermodynamic systems [*Gardiner*, 1983].

### 3.4. Experimental validation

457 Experiments were performed in a tilted, narrow, glass-sided channel, 2 m in length and  
 458 20 cm in height, as depicted in Fig. 3. The channel width  $W$  was adjusted to 6.5 mm,  
 459 which was slightly larger than the particle diameter  $2a = 6$  mm. In this way, particle

460 motion was approximately two-dimensional. The channel slope  $\tan \theta$  ranged from 7.5%  
461 to 15%.

462 The channel base consisted of half-cylinders of equal size (radius  $r = a$ ) and randomly  
463 arranged on different levels. Colored spherical glass beads with a density  $\rho_p = 2500 \text{ kg/m}^3$   
464 (provided by Sigmund Lindner GmbH, Germany) were injected from a reservoir into the  
465 channel using a wheel. For the experiments presented here, the injection rate  $\dot{n}_0$  was  
466 5–21 beads per second, with an uncertainty of less than 5%. This corresponded to a solid  
467 discharge per unit width  $q_s = Q_s/W$  of  $9\text{--}38 \times 10^{-5} \text{ m}^2/\text{s}$ .

468 The experimental conditions (velocity profile, bed friction, etc.) were specified in earlier  
469 papers [Ancey *et al.*, 2008]. Although the flume was narrow, we checked that its hydraulic  
470 characteristics were like those observed in wide channels with shallow flows [Frey and  
471 Reboud, 2001]. The flow Reynolds number was in the 3000–7700 range. The Froude  
472 number varied significantly over the experimental duration and along the main stream  
473 direction. Its mean values were close to 1.5 (in the 1-4 range), which means that the  
474 flows were supercritical on average, but its instantaneous values fluctuated a great deal  
475 and frequent transitions to subcritical regimes occurred. Note that we defined the solid  
476 discharge  $\dot{n}$  as the flux of beads per unit time. We can also define the solid discharge  
477  $Q_s$  as the volume flow rate and relate it  $\dot{n}$  using  $Q_s = \pi(2a)^3\dot{n}/6$ . For  $\dot{n}$ , we used the  
478 same definition (4) as that used in the theoretical analysis. The bottom shear stress is  
479 evaluated as  $\tau_b = \rho gh \sin \theta$  [Frey and Reboud, 2001].

480 We ran 15 experiments with different inclinations and various flow rates. Bed load  
481 equilibrium flows were achieved. The features of each run are summarized in Table 1. We

482 report the water discharge  $q_w$ , the mean flow depth  $h$ , the mean water velocity  $u_f = q_w/h$ ,  
483 the solid discharge  $\dot{n}$ . The notation E10-6 means:  $\tan \theta = 10\%$  and  $\dot{n} \approx 6$  beads/s.

484 The particles and the water stream were filmed using a Pulnix partial scan video camera  
485 (progressive scan TM-6705AN), placed perpendicular to the glass panes at 115 cm away  
486 from the channel, approximately 80 cm upstream from the channel outlet. The camera  
487 was inclined at the same angle as the channel, behind which lights were positioned. An  
488 area of  $L = 22.5$  cm in length and 8 cm in height was filmed. The camera resolution  
489 was  $640 \times 192$  pixels for a frame rate of 129.2 fps (exposure time: 0.2 ms, 256 gray  
490 levels). Each sequence comprised 8000 images because of limited computer memory;  
491 this corresponded to a duration of approximately one minute. Positions of the bead  
492 mass centers were detected by means of an algorithm combining several image-processing  
493 operations; particle trajectories were calculated using a tracking algorithm [Böhm *et al.*,  
494 2006].

Table 1: Flow characteristics and measured values.

| Experiment                           | E7-6  | E7-8  | E7-9  | E7-11 | E10-6 | E10-7 | E10-8 | E10-9 | E10-16 | E10-21 | E12-9  | E12-16 | E12-21 | E15-16 | E15-21 |
|--------------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|--------|--------|--------|--------|--------|
| number                               | (a)   | (b)   | (c)   | (d)   | (e)   | (f)   | (g)   | (h)   | (i)    | (j)    | (k)    | (l)    | (m)    | (n)    | (o)    |
| $\tan \theta$ (%)                    | 7.5   | 7.5   | 7.5   | 7.5   | 10.0  | 10.0  | 10.0  | 10.0  | 10.0   | 10.0   | 12.5   | 12.5   | 12.5   | 15.0   | 15.0   |
| $q_w$ ( $10^{-3}$ m <sup>2</sup> /s) | 10.00 | 11.54 | 13.85 | 26.15 | 4.15  | 4.42  | 5.38  | 5.54  | 8.19   | 10.31  | 2.97   | 3.85   | 4.46   | 2.31   | 2.92   |
| $h$ (mm)                             | 18.9  | 20.8  | 24.9  | 40.8  | 10.2  | 10.8  | 12.2  | 12.5  | 16.9   | 19.4   | 7.0    | 8.2    | 9.4    | 4.9    | 6.7    |
| $u_f$ (m/s)                          | 0.53  | 0.55  | 0.56  | 0.64  | 0.41  | 0.41  | 0.44  | 0.44  | 0.48   | 0.53   | 0.42   | 0.47   | 0.48   | 0.47   | 0.44   |
| $\dot{n}$ (beads/s)                  | 5.45  | 7.76  | 9.20  | 10.99 | 5.72  | 6.85  | 7.74  | 9.41  | 15.56  | 20.57  | 9.52   | 15.52  | 19.86  | 15.45  | 20.55  |
| $\bar{u}_p$                          | 0.16  | 0.17  | 0.15  | 0.15  | 0.11  | 0.13  | 0.11  | 0.12  | 0.13   | 0.12   | 0.08   | 0.09   | 0.10   | 0.07   | 0.08   |
| $N_r$                                | 4.62  | 6.01  | 8.33  | 9.96  | 8.31  | 7.51  | 10.42 | 11.35 | 18.15  | 26.55  | 20.00  | 30.15  | 33.73  | 41.87  | 45.56  |
| $N_s$                                | 2.25  | 3.22  | 3.94  | 5.21  | 2.35  | 2.99  | 3.28  | 3.85  | 6.09   | 7.74   | 1.83   | 3.09   | 4.86   | 0.84   | 3.42   |
| $\text{var} N$                       | 22.53 | 30.94 | 35.43 | 38.51 | 70.74 | 36.89 | 64.55 | 49.09 | 72.09  | 129.91 | 123.38 | 182.68 | 222.88 | 214.31 |        |
| $\sigma$ (1/s)                       | 4.67  | 5.28  | 5.04  | 4.90  | 5.12  | 4.86  | 4.94  | 4.73  | 4.52   | 4.39   | 4.83   | 4.77   | 4.51   | 4.20   | 3.90   |
| $\mu$ (1/s)                          | 3.73  | 4.14  | 3.80  | 3.46  | 4.96  | 4.07  | 4.33  | 3.91  | 3.55   | 3.66   | 4.33   | 4.32   | 4.15   | 3.64   | 2.73   |
| $\lambda$ (beads/s)                  | 6.59  | 11.21 | 15.13 | 21.40 | 1.53  | 7.79  | 8.42  | 11.23 | 22.07  | 24.77  | 10.95  | 14.64  | 13.62  | 24.49  | 55.71  |
| $\nu_{in}$ (beads/s)                 | 5.44  | 7.76  | 9.19  | 10.98 | 5.71  | 6.84  | 7.73  | 9.41  | 15.55  | 20.56  | 9.51   | 15.51  | 19.85  | 15.45  | 20.54  |
| $\nu_{out}$ (1/s)                    | 0.77  | 0.78  | 0.75  | 0.73  | 0.60  | 0.69  | 0.56  | 0.68  | 0.68   | 0.60   | 0.43   | 0.47   | 0.51   | 0.35   | 0.42   |

495 Table 1 reports the mean particle velocity in the rolling regime  $\bar{u}_r$ , the mean particle  
 496 velocity  $\bar{u}_p$ , the number of particles in a rolling regime  $N_r$  and in a saltating regime  $N_s$  (the  
 497 total number of moving particles being  $N = N_r + N_s$ ), the entrainment and deposition  
 498 rates  $\lambda$ ,  $\sigma$ ,  $\mu$  together with the inflow and outflow rates  $\nu_{in}$  and  $\nu_{out}$ . The entrainment  
 499 and deposition rates  $\mu$  and  $\lambda$  were estimated using (2) and the equilibrium conditions for  
 500 the bed  $(\sigma + \nu_{out})\bar{N} = \nu_{in} + \lambda + \mu\bar{N}$  and the control volume  $\nu_{out}\bar{N} = \nu_{in}$ , where  $\bar{N}$  is  
 501 the mean number of moving particles. The deposition rate  $\sigma$  was measured using image  
 502 processing, while  $\nu_{in}$  is imposed at the flume inlet.

503 A striking result is the weak dependence of the mean particle velocity on bottom shear  
 504 stress  $\tau_b$ . As shown in Fig. 4, for a given slope, the mean particle velocity reaches a nearly  
 505 constant level independently of  $\tau_b$  (but depending on  $\theta$ ). For instance, for  $\theta = 10\%$ , for  
 506  $\tau_b \approx 10$  Pa (run E10-6 in Table 1), the mean particle velocity is  $\bar{u}_p = 11$  cm/s against  
 507  $\bar{u}_p = 12$  cm/s for  $\tau_b \approx 20$  Pa (run E10-21 in Table 1). Using a wide flume, Schmeckle  
 508 observed the same behavior (M.W. Schmeckle, personal communication 2007), which  
 509 leads us to think that this behavior may be quite general for intermittent bedload transport  
 510 conditions. Note that this behavior was induced by the experimental setup (in particular,  
 511 the narrow flume), but a consequence of collective motion; when a single particle was  
 512 in motion in the window, the particle velocity markedly depend on fluid velocity [*Ancey*  
 513 *et al.*, 2008].

514 Interestingly enough, when we reported the emigration rate  $\nu_{out}$  as a function of the  
 515 particle velocity, we found out that  $\nu_{out} = \bar{u}_p/L'$ , with  $L' = 20$  cm, as shown in Fig. 5.  
 516 This shows that the outflow rate is directly proportional to the mean particle velocity,  
 517 which is to be expected. Indeed recall that theoretically, the particle flow rate was defined

518 as the volume average of particle velocities  $\dot{n} = N\bar{u}_p/L$  [see Eq. (4)] and that in a  
 519 steady state, the flux of outgoing particles is  $\nu_{out}N = \dot{n}$ , which gives  $\nu_{out} = \bar{u}_p/L$  under  
 520 steady state conditions. Experimentally we found that this relation holds if we replaced  
 521  $L = 22.5$  cm with  $L' = 20$  cm; this 10% deviation shows that defining the particle flow  
 522 rate as volume averages or fluxes does not lead strictly to the same values. Note that we  
 523 also ran experiments with various  $L$  values; measuring the intercorrelation functions of  
 524  $N(t)$  measured in neighboring windows confirmed that  $\nu_{out}$  is related to particle velocity  
 525 [Böhm *et al.*, 2004].

526 We can compare the approximate solution (34) with the exact solution (1) to the master  
 527 equation (20) for steady state conditions. For large systems ( $L \rightarrow \infty$ ), we have  $\alpha =$   
 528  $\sigma + \nu_{out} \rightarrow \sigma$  while  $\beta = \lambda + \nu_{in} \rightarrow \lambda$ , which means that the variances and means of  
 529 both distributions become closer. We retrieve the fact that in the large-system limit, a  
 530 negative binomial distribution can be approximated by a normal distribution, with the  
 531 only difference that normal distributions are continuous real-valued functions (which can  
 532 then take negative values) whereas negative binomial distributions are functions that take  
 533 nonnegative integer values. Note also that when  $\mu > 0$  (collective entrainment), the ratio  
 534  $\text{var}N/\bar{N}$  always exceeds unity, which implies that large and frequent fluctuations occur.  
 535 When  $\mu = 0$  (no collective entrainment), this ratio is unity and we retrieve the usual  
 536 Poissonian limit of Einstein-like models. The auto-correlation time of the exact solution  
 537 (1) to the master equation (20) is  $t_c = (\alpha - \mu)^{-1}$  [Ancey *et al.*, 2008]; in the large-system  
 538 limit, we also find that the relaxation time approaches the autocorrelation time:  $t_r \rightarrow t_c$ .

539 Figure 6 shows the probability distribution function of the number of moving particles  
 540  $N$ . We report the empirical distribution (dots), the exact theoretical solution (1) (dashed

541 line), and the approximate analytical solution (34) (solid line). The parameters of the  
542 distributions were estimated using the measured values  $\text{var}N$  and  $\bar{N}$  reported in Table 1;  
543 for the sake of readability, we plotted the discrete probability mass functions as continuous  
544 curves. First note the fairly good agreement between the data and theoretical probability  
545 distribution for all experiments. Local departures and data scattering are seen, but they  
546 are usually associated with low probabilities. Since 8000 data points were used for each  
547 run, empirical probabilities lower than  $10^{-3}$  are unimportant. The plots confirm that the  
548 negative binomial distribution is a good candidate for describing the statistical behavior  
549 of  $N$  over a wide range of flow conditions.

550 There is a significant change in the distribution shape when the water discharge in-  
551 creases. When it is low [e.g., panel (e) in figure 6], the empirical distribution is close to  
552 a straight line on a logarithmic scale, implying that the number of particles decays ex-  
553 ponentially within the observation window. For experiment (e), the discrepancy between  
554 empirical and theoretical probability distributions is the most pronounced of all runs,  
555 which may mean that theory performs less well for these low discharges. With increasing  
556 water discharge, the probability distribution takes a bell shape, which is first skewed, then  
557 nearly symmetric at the highest slopes and flow rates [see panel (n) of figure 6]. In the  
558 case of run (n), the discrete empirical probability distribution can be closely approximated  
559 by a Gaussian distribution, as expected.

560 For each run, the sample variance exceeded the mean, but the ratio  $\text{var}N/\bar{N}$  decreased  
561 with increasing water discharge. For instance, for slope  $\tan\theta = 0.1$ , the ratio  $\text{var}N/\bar{N}$   
562 was as high as 7 for run (e) and dropped to below 3 for run (o). Thus for each run,  
563 the number of moving particles varied frequently and widely, but with increasing water

564 discharges, wide fluctuations became less frequent, which substantiates the idea that at  
 565 high water discharges, sediment transport becomes more continuous.

566 The normal distribution closely approximates the empirical distribution of  $N$  when  $N$   
 567 is sufficiently large (typically for  $\bar{N}$  as large as 40). There is, however, no precise criterion  
 568 that can serve to mark the lower bound from which this approximation holds.

### 3.5. Implication for the Exner equation

569 The stochastic Exner equation is given by (19), where the entrainment and deposition  
 570 parameters  $\sigma$ ,  $\mu$ , and  $\tilde{\lambda}$  depend on the bottom shear stress (15) while the number of moving  
 571 particles  $\tilde{N}$  is the solution to the Ito differential equation (27). The full problem (19),  
 572 (15), and (27) is prohibitively complex to be solved analytically because of nonlinearities  
 573 arising in the coupled equations and stochastic fluctuations. Analytical approximations  
 574 are available if we assume that the flow depth is nearly constant, an assumption which is  
 575 quite common for planar-bed lower-regime flow conditions [*Exner, 1925; Jerolmack and*  
 576 *Mohrig, 2005*]. In that case, the bottom-shear-stress equation is constant:  $\tau_b = \rho gh \sin \theta$ .

577 Different levels of approximation can be used to shed light on the behavior of the system  
 578 of equations (19) and (27). The crudest approximation is to assume that  $N$  varies so fast  
 579 with time that for periods of time longer than the relaxation time  $t_r$ , the number of  
 580 moving particles reaches a quasi-stationary state in which  $dN/dt \rightarrow 0$  (or  $d\eta_1 \rightarrow 0$ ). In  
 581 this case, using (28) and (32), we obtain

$$\tilde{N} = \frac{\tilde{\lambda}}{\sigma - \mu} + \frac{\sqrt{\tilde{B}(n_*)} d\mathcal{W}}{\sigma - \mu} dt, \quad (35)$$

582 which results in

$$b(x, t) = \sqrt{\tilde{B}(n_*)} \mathcal{W}(t). \quad (36)$$



583 This equation shows that the probability distribution function of bed elevation  $b$  is a  
 584 Gaussian distribution; furthermore, the variance of  $b$  grows linearly in time since the  
 585 variance of  $\mathcal{W}$  grows like  $t$ . A Gaussian distribution has often been found to be a correct  
 586 approximation of bed elevation profiles [Nikora *et al.*, 1998]. Experimental measurements  
 587 provided variance growing as a power function of time, with an exponent ranging from  
 588 0.56 to 1.36 depending on particle size [Coleman *et al.*, 2005]. In spite of its crudeness,  
 589 this approximation provides a reasonably correct description of the bedform evolution for  
 590 planar-bed incipient-motion flow conditions.

591 In the large-system limit ( $L \gg 1$ ) and slightly bumpy bed ( $\partial_x b \ll \tan \theta$ ), another  
 592 approximation consists in expanding  $b$  into an  $\epsilon$ -power series as we did for  $N$ :  $b(x, t) =$   
 593  $b_0(x, t) + \epsilon b_1(x, t) + \dots$ . We then obtain a hierarchy of equations, where the exchanges  
 594 rates  $\sigma$ ,  $\mu$ , and  $\tilde{\lambda}$  are constant. To zeroth order, we obtain

$$\psi \frac{\partial b_0}{\partial t} = (\sigma - \mu) \tilde{n}_* - \tilde{\lambda}, \quad (37)$$

595 Since  $N_* \rightarrow N_\infty = \tilde{\lambda}/(\sigma - \mu)$  when  $t \rightarrow \infty$ , then we get  $\partial_t b \rightarrow 0$ , which means that at  
 596 order  $\epsilon^0$ , the bed elevation reaches a constant level. To first order, we obtain

$$\psi \frac{\partial b_1}{\partial t} = (\sigma - \mu) \eta_1, \quad (38)$$

597 with  $\eta_1$  solution to (32) subject to the initial condition  $\eta_1(0) = 0$ :

$$\eta_1(t) = \int_0^t \exp(-(t-t')/t_r) \tilde{B}^{1/2}(n_*(t')) d\mathcal{W}(t'), \quad (39)$$

598 from which we deduce that

$$b_1(x, t) = (\sigma - \mu) \int_0^t dt' \int_0^{t'} \exp(-(t'-t'')/t_r) \tilde{B}^{1/2}(n_*(t'')) d\mathcal{W}(t''), \quad (40)$$

599 At long times, we can compute the first moments of  $b_1$ :  $\langle b_1 \rangle = 0$  and  $\langle b_1^2 \rangle = \tilde{B}(N_\infty)t$ ,

600 which is consistent with the crude approximation (36). In particular, we retrieve that the

601 variance varies linearly with time on the long range. Another interesting point is related  
 602 to the spectral density function  $\mathcal{S}(\omega)$  of  $\tilde{n}_1$ : since on the long term  $\eta_1$  is an Ornstein-  
 603 Uhlenbeck process with constant parameter  $k_o = \sigma - \mu$  and diffusion  $D_o = \tilde{B}(n_\infty)$ , the  
 604 spectrum of  $\tilde{N}$  is [*Gardiner*, 1983]

$$\mathcal{S}(\omega) = \frac{\tilde{B}(N_\infty)}{2\pi} \frac{1}{\omega^2 + (\sigma - \mu)^2}, \quad (41)$$

605 with  $\omega$  the angular frequency. This shows that the spectrum tends toward a plateau when  
 606  $\omega \rightarrow 0$  and decreases as  $\omega^{-2}$  in the limit of  $\omega \rightarrow \infty$ . The bed elevation following the same  
 607 trend, we retrieve the  $\omega^{-2}$  decay observed by *Nikora et al.* [1997] in the high-frequency  
 608 range, but the low-frequency behavior  $\mathcal{S}(\omega) \propto \omega^{-3}$  is not retrieved. As discussed by  
 609 *Nikora et al.* [1997], note that there is no consensus in the literature about the decay rate  
 610 of  $\mathcal{S}$  in the high-frequency domain (some authors providing  $\mathcal{S} \propto \omega^{-3}$ ).

611 The last point concerns the variation in the root mean square of bed elevation  $w$  with  
 612 length  $L$ . Since  $b = b_0 + \epsilon b_1 + \dots$  with  $\epsilon = L^{-1/2}$ ,  $\langle b_1 \rangle = 0$ , and  $\langle b_1^2 \rangle = \tilde{B}(N_\infty)t$ , we have

$$w(L, t) = \left( \frac{1}{k} \sum_{i=1}^k (b(x_i, t) - \bar{b})^2 \right)^{1/2} = \epsilon \left( \frac{1}{k} \sum_{i=1}^k b_1^2(x_i, t) \right)^{1/2}, \quad (42)$$

613 where  $b(x_i, t)$  denotes the bed elevation at a site  $x_i$  and time  $t$ ,  $k$  is the number of sites over  
 614 the length  $L$ , and  $\bar{b}$  is the average of  $b$  over the domain of length  $L$ . The sum of  $k$  square  
 615 Gaussian variables of mean 0 and constant variance (at time  $t$ ) is a (noncentral) chi-  
 616 square distribution with mean  $k$ . Since  $k \propto L$ , we then deduce that  $w(L, t) \propto \epsilon L = L^{1/2}$ .  
 617 This is roughly consistent with the results obtained by *Jerolmack and Mohrig* [2005], who  
 618 obtained  $w \propto L^{0.64}$ ; although there is no perfect match, both values are reasonably close  
 619 (given that the data span over a limited range of variations).

### 3.6. Numerical simulations

Two types of numerical models can be developed depending on whether  $N$  is considered a continuous or discrete random variable. The former type is the easiest to implement since it involves solving a set of coupled first-order differential equations including a stochastic differential equation; there are many relatively simple, but fairly efficient algorithms for that purpose [Higham, 2001]. The latter type is a little more delicate to use in that it is based on exact or approximate stochastic simulation algorithms, whose running is known to be fairly slow [Gillespie, 2007].

Here we address the case in which the flow depth is nearly constant and  $\tilde{N}$  is considered a continuous variable. We have to solve (19), (15), and (27). We use a first-order explicit forward Euler scheme (a variant called Euler-Maruyama method) to discretize the governing equations with a constant time step  $\Delta t$  [Iacus, 2008]. The spatial domain of length  $L$  is divided into  $S$  cells centered at nodes  $x_i = (i - \frac{1}{2})\Delta x$  ( $1 \leq i \leq S$ ) of size  $\Delta x$ . The bed elevation is evaluated at each node  $x_i$  and time  $t_k = k\Delta t$ , where  $\Delta t$  is a very small time increment. At time  $t_{k+1} = (k + 1)\Delta t$ , we have

$$\tilde{N}_{i,k+1} = \tilde{N}_{i,k} + \Delta t \tilde{A}(\tilde{N}_{i,k}) + \zeta \sqrt{\Delta t \tilde{B}(\tilde{N}_{i,k})}, \quad (43)$$

$$b_{i,k+1} = b_{i,k} + \frac{\Delta t}{\psi} \left( (\sigma + \nu_{out} - \mu) \tilde{N}_{i,k} - \tilde{\lambda}(\tau_{i,k}) \right). \quad (44)$$

where  $\zeta$  is picked from a Gaussian distribution with mean 0 and variance 1. This scheme is first-order in time and requires tiny time steps to be accurate; there are other methods that perform better and converge faster [Rümelin, 1982]. Typically, with the Euler-Maruyama scheme, the simulation of a 60-s run requires computational times as long as 6 hours on a standard personal computer.

To illustrate this scheme, we present the results of a numerical simulation for a virtual 25-m long flume tilted at  $\theta = 5^\circ$  to the horizontal. The parameters of the simulation are

641 reported in the caption of Fig. 7. As shown in the Fig. 7 a, the solid concentration varies  
642 substantially along the bed, causing large variations in the bed elevation (see Fig. 7 b).  
643 The empirical distribution of the simulated  $N$  samples are nearly Gaussian (see Fig. 7 d),  
644 while the bed-elevation sample is also Gaussian (see Fig. 7 c) with a variance that grows  
645 nearly linearly with increasing time (see Fig. 7 e). These results are in good agreement  
646 with the analytical approximations seen in §3.2.

#### 4. Concluding remarks

647 In this paper we have built a reasonably simple framework for simulating the bed  
648 evolution of a river bed made up of coarse sediment when particles move sporadically  
649 along the bed. In a first-order approximation of lower-regime flow conditions, we ignore  
650 the details of fluid flow and focus our attention on the sediment-stream interface. The  
651 model is thus a first stab at modelling bedload transport when there is no significant  
652 feedback from water flow.

653 Our posit is that for bottom shear stresses just in excess of the threshold of incipi-  
654 ent motion (i.e. for transport-stage parameters  $T < 3$ ), sediment flux is driven by the  
655 concentration in moving particles rather than particle velocity (which varies weakly with  
656 increasing bottom shear stress). Therefore, to compute the solid discharge or track the  
657 bed evolution, it is fundamental to predicting how the number of moving particles varies  
658 with time. With this objective in mind, we idealized particle entrainment and deposition  
659 as population exchanges between the stream and the bed inside a finite control volume.  
660 We used an immigration–emigration birth–death Markov process to theoretically describe  
661 the probabilities that particles are entrained from or deposited on the bed, while other  
662 particles may enter or leave the control volume. In an earlier paper [*Ancey et al.*, 2008],

663 we showed that under steady state conditions, the probability distribution of the number  
664 of moving particles is a negative binomial distribution. This distribution has interesting  
665 properties such as its wide variance, which may explain the huge fluctuations experienced  
666 by sediment flux, and thus is an interesting alternative to more elaborated formalisms  
667 (based on fractional diffusion) presented in this special issue. In this paper, we extend  
668 theory by considering the limit of large systems. When the size of the control volume  
669 or the number of moving particles becomes large (typically  $\bar{N}/L > 2$  particles/cm), we  
670 show that the concentration in moving particles behaves like a Gaussian random variable.  
671 This approximation is here particularly helpful in that it transformed the Exner equation  
672 (14) into an Ito stochastic differential equation (14). Some interesting approximations  
673 can be directly inferred from this equation. In particular, under planar-bed lower-regime  
674 conditions, it can be shown that the bed-elevation density distribution is Gaussian, with a  
675 variance growing linearly with time. Both analytical approximations and numerical sim-  
676 ulations support the idea developed by *Jerolmack and Mohrig* [2005] according to which  
677 accounting for fluctuations of sediment flux in the Exner equation is crucial to obtaining  
678 bed topographies comparable to natural river beds.

679 There are several related topics that would deserve further attention. First, as noted  
680 in §A, there are several ways of defining the solid discharge and it is currently quite  
681 unclear whether these definitions somehow overlap and match. In particular, the statistical  
682 properties of the solid discharge depend a great deal on the definition used. A point that  
683 requires more consideration is the relation between the solid discharge  $q_s$ , mean particle  
684 velocity  $\bar{u}_p$ , and particle concentration  $\tilde{N}$ . Here, relying on laboratory observations, we  
685 assumed that much of the  $q_s$  variance is due to variations in the particle concentration

rather than particle velocity; this contrasts with what is assumed in most theoretical bedload transport models. Second, the statistics of bedload transport is substantially influenced by the scale of observation (i.e., the length  $L$  of the control volume), with the important consequence that the bed-elevation features are also scale dependent, as noted by *Jerolmack and Mohrig* [2005]. For short spatial scales, the number of moving particles resembles a discrete and random process, while for long scales, the behavior is more continuous and the system exhibits Gaussian fluctuations. Note that even on small spatial scales ( $L \approx 40a$ , with  $a$  the particle radius), the particle concentration can exhibit Gaussian behavior when the particles move in large numbers in the control volume. Third, in contrast with solute transport in porous media (see the discussion in § 2.2), there is still no clear connection between Eulerian and Lagrangian descriptions, which would be essential for both theoretical and practical reasons.

A strength of the theoretical model is its versatility, which should allow consideration of scenarios much closer to reality than the one briefly presented in § 3.6 and extension to similar problems, such as long-term behavior of rivers [*Tipper, 2007*] and landforms [*Turcotte, 1995*]. The model offers a simple and efficient way of describing complex issues in sediment transport by avoiding some of the impediments encountered with other approaches. For instance, detailed experimental investigations have revealed that instantaneous forces experienced by coarse particles lying on the bed are not strongly correlated with the mean velocity field of the flow [*Schmeeckle et al., 2007*]. This suggests that a fully deterministic approach to sediment transport remains a formidable challenge; our hybrid stochastic-physical approach tries to get around this difficulty by substituting the complex coupling between flow and sediment with a statistical description. Another great

709 advantage is that all parameters used in the model are physical quantities that can be  
710 fairly easily measured in the laboratory or in the field using image processing [*Drake et al.*,  
711 1988; *Ancey et al.*, 2008]. There are a few limitations of this approach that come out.  
712 First, when there is no coupling between the water flow and sediment transport (as as-  
713 sumed in the numerical simulations presented in § 3.6), the model predicts bed roughness  
714 growing like  $t^{1/2}$ . There is a naturally an upper bound to this growth imposed by the  
715 water flow [*Coleman et al.*, 2005], in particular by coherent turbulent structures that are  
716 generated by bedforms, interfere with the bed interface, and affect turbulence produc-  
717 tion/dissipation processes [*Best*, 2005]. Second, numerical methods for solving stochastic  
718 problems require considerable amounts of computational times and are not free of pitfalls.  
719 In particular, there is no available analytical solution against which numerical models can  
720 be checked.

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## Appendix A: Computation of the solid discharge

733 There is no unique way to define the solid discharge. The random and time-evolving  
 734 nature of solid discharge raises the problem of a suitable statistical procedure that enables  
 735 computation and description of sediment flux. In spite of valuable efforts in recent years  
 736 to gain insight into this issue, it is still unclear whether the different solid-discharge forms  
 737 lead to compatible results in terms of statistical properties. Here we review the main  
 738 forms that are currently of common use and comment on their strengths and weaknesses.

739 While theoretically the volume solid discharge can always be defined as the flux of  
 740 particles through a cross-section  $S$  of unit width:

$$q_s = \int_S \mathbf{u}_p \cdot \mathbf{k} dS, \quad (\text{A1})$$

741 with  $\mathbf{u}_p$  the particle velocity field and  $\mathbf{k}$  the unit normal to  $S$ , this definition is rarely used  
 742 in practice as it is more suited to continuous fields than discrete elements; the dimension  
 743 of  $q_s$  is  $\text{m}^2 \text{s}^{-1}$ . At least four different forms of the solid discharge have been proposed  
 744 for practical or theoretical purposes. They all assume bedload at equilibrium or near  
 745 equilibrium. On average, they may provide the same values, but the statistical properties  
 746 of  $q_s$  are influenced a great deal.

747 A variant of equation (A1) is to count the number of particles that passed through  $S$   
 748 over a short time increment  $\delta t$ . The main problem is that  $q_s$  is a step function, which takes  
 749 zero values except at the times of arrival of individual particles; the resulting signal is  
 750 then very noisy. In practice, using hydrophones or piezoelectric sensors makes it possible  
 751 to evaluate  $q_s$  by correlating acoustic signals (or vibrations) and sediment rate [*Bänziger*  
 752 *and Burch*, 1991; *Carling et al.*, 1998; *Tunnickliffe et al.*, 2000; *Rickenmann and McArdell*,  
 753 2007; *Rickenmann*, 2008; *Turowski et al.*, 2008]. Another related form is to count the

754 number of particles that arrive up to time  $t$  or to integrate  $q_s$  over a short period of  
 755 time. In the laboratory, this is done by weighting the material accumulated in a basket  
 756 located at the flume outlet while in the field, sediment traps and bedload samplers are  
 757 used. These techniques do not provide  $q_s$  directly, but the sediment volume per unit width  
 758  $V(t) = \int q_s(t)dt$ . In principle, it should be possible to differentiate  $V$  to derive  $q_s$ , but  
 759 in practice, fluctuations in the  $V(t)$  records make this operation delicate, which explains  
 760 why sampling time is a key issue when trying to properly evaluate the solid discharge  
 761 [*Bunte and Abt, 2005; Singh et al., 2009*].

762 Another approach was taken by *Einstein* [1950] according to whom sediment transport  
 763 does not result from an equilibrium in the momentum transfers between solid and liquid  
 764 phases, but rather from the difference between the entrainment and deposition rates,  $E$   
 765 and  $D$ , respectively, which are functions of the flow conditions and bed geometry [*Einstein,*  
 766 1950]. This amounts to writing that on a small interval  $\Delta x$ , the solid discharge variation  
 767 is  $\delta q_s = (E - D)\Delta x$ , and so the solid discharge at bed equilibrium is the implicit solution  
 768 to the equation  $E = D$ . Einstein found that the sediment transport rate is

$$q_s = E\ell_s, \quad (\text{A2})$$

769 where  $\ell_s$  is the mean length travelled by individual particles during each step. Several  
 770 field measurement campaigns used Einstein-like definitions to monitor bedload transport  
 771 under incipient-motion conditions [*Drake et al., 1988; Habersack, 2001; Wilcock, 1997;*  
 772 *Pryce and Ashmore, 2003*]. Laboratory experiments as well as theoretical analyses made  
 773 extensive use of this definition [*Fernandez Luque and van Beek, 1976; Seminara et al.,*  
 774 2002]. The variance of  $q_s(t)$  records results from both fluctuations in entrainment rates  $E$   
 775 and heap lengths  $\ell_s$ .

776 The use of tracer stones in gravel-bed rivers has given rise to a third relation. From the  
 777 observation that particles can be moving, lying at rest on the bed surface, or buried in  
 778 the bed, one can define a virtual velocity, which is the time-averaged velocity  $U_p$  (called  
 779 virtual particle velocity) of a single particle regardless of its state. Only the upper bed layer  
 780 participates in bedload transport and is therefore termed the active layer; the thickness of  
 781 this layer is  $L_a$  and represents the depth down to which the bed is continuously reworked  
 782 by fill and scour. Mass conservation then implies that

$$q_s = U_p L_a. \quad (\text{A3})$$

783 The use of this equation has been documented for both natural rivers [*Ferguson et al.*,  
 784 2002; *Ferguson and Hoey*, 2002; *DeVries*, 2002; *Cudden and Hoey*, 2003] and flume exper-  
 785 iments [*Wong et al.*, 2007]. The statistical properties of  $q_s(t)$  depend on the fluctuations  
 786 of  $U_p$  and  $L_a$ , which are little known in practice.

787 Taking inspiration from kinetic theories of gas and two-phase flow rheology [*Drew and*  
 788 *Passman*, 1999; *Lhuillier*, 1992], we can define the solid discharge as the ensemble average  
 789 of the particle flux

$$\langle q_s \rangle = \int_S \int_{\mathbb{R}^2} P[\mathbf{u}_p \mid \mathbf{x}, t] \mathbf{u}_p \cdot \mathbf{k} |d\mathbf{x}| d\mathbf{u}_p. \quad (\text{A4})$$

790 where  $P[\mathbf{u}_p \mid \mathbf{x}, t]$  is the probability that a particle crosses the control surface  $S$  at  
 791 position  $\mathbf{x}$  and time  $t$  with velocity  $\mathbf{u}_p$ . Note that  $\mathbf{u}_p$  is a velocity field, which is  $\mathbf{u}_p(\mathbf{x}, t) =$   
 792  $\mathbf{u}_G + \boldsymbol{\Omega} \times (\mathbf{x} - \mathbf{x}_G)$  where  $\mathbf{x}$  lies inside a particle with velocity  $\mathbf{u}_G$  at the center of mass  
 793  $\mathbf{x}_G$  and rotational velocity  $\boldsymbol{\Omega}$ ; when  $\mathbf{x}$  lies in the fluid phase, this field is zero. Figure 8  
 794 shows a particle crossing the control  $S$ .

795 Under near-equilibrium conditions, ensemble averages can be swapped with volume  
 796 averages; equation (A4) can then be recast in the following form

$$\langle q_s \rangle = \int_b^s c(y) u_p(y) dy, \quad (\text{A5})$$

797 where we retrieve the correspondence between concentration and probability of finding  
 798 one particle at a given place (see §2.2). This equation was used notably by *Wiberg*  
 799 *and Smith* [1989]. Similar expressions were also used by *Bridge and Dominic* [1984],  
 800 *Yalin* [1963], and *Kovacs and Parker* [1994]. The equivalence between ensemble and  
 801 volume averages makes sense when the entire flow is homogeneous, i.e., when the particles  
 802 are homogeneously distributed in the streamwise direction. For two-phase flows over  
 803 mobile beds, bedforms usually affect the distribution of moving particles, which makes  
 804 the assumption of homogeneity dubious. A number of tricks have been developed to  
 805 extend ensemble and volume average techniques to non-uniform particle suspensions; for  
 806 instance, an infinite uniform suspension can be approximated by the periodic repetition of  
 807 a control volume. Numerically, this amounts to carrying out the simulation in a finite box  
 808 with periodic boundary conditions [*Marchioro et al.*, 2000]. It is convenient to introduce  
 809 a finite-volume average solid discharge

$$\langle q_s \rangle_L = \frac{S}{V} \sum_{i=1}^N u_i v_p = \frac{v_p}{L} \sum_{i=1}^N u_i, \quad (\text{A6})$$

810 where  $v_p$  is the particle volume and integration has been performed over the control volume  
 811 ( $\mathcal{V} = L \times S$ ) of length  $L$ , which is sufficiently long to contain a number of particles, but  
 812 is short enough compared to the scale of variation of  $q_s$  on the macroscopic scale. We let  
 813  $u_i = \mathbf{u}_p \cdot \mathbf{k}$  denote the streamwise velocity component of particle  $i$ . Here we thus define

814 the particle flow rate  $\dot{n} = Q_s/v_p$  as

$$\dot{n} = \frac{1}{L} \sum_{i=1}^N u_i = \frac{N}{L} \bar{u}_p, \quad (\text{A7})$$

815 where  $\bar{u}_p$  denotes the mean particle velocity.

## Appendix B: Birth-death model

816 To begin with, we set the scene by developing an analogy between chemical reactions  
817 and particle transport in streams. This makes it possible to use many of the computational  
818 tools developed in chemical kinetics. We then outline the stochastic basis of the model.

819 As depicted in Fig. 1, we consider a control volume  $\mathcal{V}$  (per unit width) within which we  
820 track moving particles and record deposition/entrainment events. A few solid particles  
821 are entrained by the water stream: they can roll/slide along the bed or they can leap  
822 and stay in saltation in the water stream for short time periods. We refer to the former  
823 motion as the *rolling regime* and to the latter as the *saltating regime*. As we are especially  
824 interested in low bedload transport, we stress flows with a fairly low fluid velocity: the  
825 trajectory of a single particle then exhibits a succession of rests and moves in a rolling or  
826 saltating regime. Below we do not discriminate between rolling and saltation and treat  
827 both motions as a single “chemical” species which we call the moving particles (M). We  
828 assume that the number of particles (B) making up the bed is infinite, i.e., whenever a  
829 particle at the bed interface is set in motion, the shape of the interface is altered, but not  
830 the number of particles available to entrainment at the bed interface.

831 Let us write the exchanges between the bed and the stream like chemical equations:

- 832 • A moving particle  $M$  can come to rest (at rate  $\sigma_0$ ) and thus be transformed into a  
833 bed particle  $B$



- 834 • The trajectory of a moving particle  $M$  can be disturbed by a bed particle, trans-  
835 forming  $M$  into a bed particle  $B$  (at rate  $\sigma_1$ )



- 836 • A moving particle can interfere with a bed particle and entrain it (at rate  $\mu$ )



- 837 • A bed particle can be entrained by the fluid and set in motion (at rate  $\lambda$ )



838 Pursuing our analogy with chemical systems, we denote the mean number of moving  
839 particles by  $[M]$ .  $B$  is like a catalyst: it is needed in the transformation, but its mean  
840 number does not vary with time, i.e.,  $[B]$  is fixed. The resulting reaction-rate equation is

$$\frac{d}{dt}[M] = \lambda + \mu[M] - \sigma[M], \quad (\text{B5})$$

841 with  $\sigma = \sigma_0 + \sigma_1$ . As seen in (B5), we have no means of differentiating the different  
842 deposition processes  $\sigma_1$  and  $\sigma_0$ , whereas entrainment includes both individual entrainment  
843 events ( $\lambda$ ) and collective effects ( $\mu$ ), a point that is essential to explaining the appearance  
844 of wide fluctuations, i.e. non-Gaussian fluctuations of sediment transport rate [*Ancey*  
845 *et al.*, 2008]. We can now place this model on a firmer theoretical ground by describing  
846 the statistics of particle exchanges. Note also that since we are counting particles in a

finite and open control window  $\mathcal{V}$ , we must take into account that particles can enter and leave  $\mathcal{V}$  at any time. Inflow and outflow rates are denoted by  $\nu_{in}$  and  $\nu_{out}$ , as shown in Fig. 1. An earlier paper gave further information on the physical mechanisms [Ancey *et al.*, 2008].

The model we have developed belongs to the class of Markov process with discrete states in continuous times since  $N(t)$  is an integer-valued function of time; more precisely, we describe sediment transport using an *emigration-immigration birth-death process* [Cox and Miller, 1965, see chap. 5]. We consider the following exchanges over the time increment  $\Delta t$ , which is assumed to be sufficiently small that two events cannot occur in  $(t, t + \Delta t)$ .

1. Beads enter the window at rate  $\nu_{in}$  (immigration). The probability that the number of particles in  $\mathcal{V}$  is incremented by one is then

$$P(n \rightarrow n + 1; \Delta t) = \nu_{in}\Delta t + o(\Delta t). \quad (\text{B6})$$

As explained in § 2, the probability that more than one particle arrive at the same time or within the time interval  $[t, t + \delta t)$  is zero in birth-death models, which means physically that the particle flow is rather dilute and that sufficiently short time increments can be selected for this condition to be satisfied.

2. Moving particles leave the window independently at rate  $\nu_{out}$  (emigration). Since there are  $n$  particles inside  $\mathcal{V}$ , the transition probability is

$$P(n \rightarrow n - 1; \Delta t) = n\nu_{out}\Delta t + o(\Delta t). \quad (\text{B7})$$

3. Two processes enable entrainment of particles from the bed (birth): a particle can be dislodged from the bed by the water stream at rate  $\lambda > 0$ ; or a moving particle can destabilize a stationary one and set it moving. This occurs at rate  $\mu$  for any moving

867 particle within the observation window. The corresponding transition probabilities are  
868 respectively

$$P(n \rightarrow n + 1; \Delta t) = \lambda_1 \Delta t + o(\Delta t), \quad P(n \rightarrow n + 1; \Delta t) = \mu n \Delta t + o(\Delta t). \quad (\text{B8})$$

869 4. A moving particle can come to rest within the window, independently at rate  $\sigma$  for  
870 each moving particle (death). The transition probability is thus

$$P(n \rightarrow n - 1; \Delta t) = n \sigma \Delta t + o(\Delta t). \quad (\text{B9})$$

871 With these assumptions and the discrete Chapman-Kolmogorov equation

$$P(n; t + \Delta t) = \sum_{-\infty}^{+\infty} P(n + i; t) P(n + i \rightarrow n; \Delta t), \quad (\text{B10})$$

872 we obtain a set of equations [Ancey *et al.*, 2008]

$$P(n; t + \Delta t) = \alpha(n + 1) \Delta t P(n + 1; t) + \quad (\text{B11})$$

$$P(n - 1; t) \{ \beta + (n - 1) \mu \} \Delta t + P(n; t) \{ 1 - \Delta t (\beta + n \alpha + n \mu) \} + o(\Delta t),$$

873 for  $n = 1, 2, \dots$ , and

$$P(0; t + \Delta t) = \alpha P(1; t) \Delta t + P(0; t) (1 - \beta \Delta t) + o(\Delta t), \quad (\text{B12})$$

874 for  $n = 0$ , with the short-hand notation  $\alpha = \sigma + \nu_{out}$  and  $\beta = \lambda + \nu_{in}$ . On rearranging  
875 the terms and letting  $\Delta t \rightarrow 0$ , we obtain the master equation

$$\frac{\partial}{\partial t} P(n; t) = (n + 1) \alpha P(n + 1; t) + (\beta + (n - 1) \mu) P(n - 1; t) - (\beta + n(\alpha + \mu)) P(n; t), \quad (\text{B13})$$

876

$$\frac{\partial P(0, t)}{\partial t} = \alpha P(1; t) - \beta P(0; t). \quad (\text{B14})$$

877 At time  $t = 0$ , there are  $N = N_0$  particles within the control volume, so we set

$$P(n; 0) = \delta(n - N_0), \quad (\text{B15})$$

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878 where  $\delta$  is the Kronecker delta function.



## Nomenclature

## 879 Roman symbols

| Variable           | Meaning  |
|--------------------|--|
| $A$                | drift function, see Eq. (20)   |
| $a$                | particle radius  |
| $b$                | elevation of the bed surface   |
| $B$                | diffusion function, see Eq. (20)                                       |
| $c$                | concentration  |
| $d$                | distance travelled by particles  |
| $D_o$              | diffusion coefficient in the Ornstein-Uhlenbeck process                |
| $D$                | deposition rate in $\text{m}^3 \text{s}^{-1}$                          |
| $E$                | entrainment rate in $\text{m}^3 \text{s}^{-1}$                         |
| $g$                | gravity acceleration $g = 9.81 \text{ m s}^{-2}$                       |
| $h$                | flow depth   |
| $K$                | diffusion coefficient in $\text{m}^2 \text{s}^{-1}$                    |
| $k$                | dummy variable   |
| $\mathbf{k}$       | unit vector  |
| $k_o$              | parameter in the Ornstein-Uhlenbeck process                            |
| $L$                | window length  |
| $\ell_s$           | heap length  |
| $N$                | number of particles within the window                                  |
| $n$                | random number of particles   |
| $N_*$              | sure function, see Eq. (30)  |
| $N_0$              | initial number of particles within the window                          |
| $\bar{N}$          | mean number of particles within the window                             |
| $\dot{n}$          | instantaneous particle flow rate in beads/s, see Eq. (4)               |
| $\tilde{N}$        | linear density of particles in beads/m                                 |
| $\tilde{N}_\infty$ | asymptotic value of the linear density                                 |
| $\mathcal{N}$      | Gaussian distribution  |
| $p$                | parameter of the negative binomial distribution, see Eq. (1)           |
| $P$                | probability density function   |
| $\bar{Q}_s$        | mean volume particle flow rate in $\text{m}^3 \text{s}^{-1}$           |
| $Q_s$              | volume particle flow rate in $\text{m}^3 \text{s}^{-1}$                |
| $q_s$              | volume particle flow rate per unit width in $\text{m}^2 \text{s}^{-1}$ |
| $q_w$              | water flow rate per unit width in $\text{m}^2 \text{s}^{-1}$           |
| $r$                | parameter of the negative binomial distribution, see Eq. (1)           |
| $s$                | elevation of the free surface  |
| $\mathcal{S}$      | spectrum, see Eq. (41)   |

| Variable      | Meaning   |
|---------------|---|
| $S$           | number of cells, see § 3.6                          |
| $S$           | cross section of the control volume, see Appendix A |
| $t$           | time  |
| $T$           | transport stage parameter $T = \tau_b/\tau_c - 1$   |
| $t_c$         | autocorrelation time, see Eq. (3)                   |
| $t_m$         | characteristic time of motion, see Eq. (3)          |
| $t_r$         | relaxation time, see Eq. (30) and § 3.3             |
| $u_f$         | fluid velocity in $\text{m s}^{-1}$                 |
| $\bar{u}_p$   | mean particle velocity in $\text{m s}^{-1}$         |
| $u_p$         | particle velocity in $\text{m s}^{-1}$              |
| $v_p$         | particle volume in $\text{m}^3$                     |
| $\mathcal{V}$ | control volume                                      |
| $\mathcal{W}$ | Wiener process                                      |
| $W$           | flume width   |
| $w$           | root mean square of the bed elevation, see Eq. (42) |
| $x$           | streamwise coordinate                               |
| $y$           | normal coordinate                                   |

### Greek and compound symbols

| variable            | meaning  |
|---------------------|--|
| $\alpha$            | shorthand notation for $\alpha = \sigma + \nu_{out}$                         |
| $\beta$             | shorthand notation for $\beta = \lambda + \nu_{in}$                          |
| $\delta$            | Kronecker symbol   |
| $\delta t$          | short time span  |
| $\Delta t$          | time increment   |
| $\Delta x$          | space increment  |
| $\epsilon$          | small parameter $\epsilon = L^{-1/2}$  |
| $\eta_1$            | first-order term in the expansion (28)                                       |
| $\gamma$            | fractional exponent in Eq. (12)  |
| $\Gamma$            | gamma function   |
| $\lambda$           | entrainment rate in beads/s  |
| $\tilde{\lambda}$   | entrainment rate per unit length in $\text{beads m}^{-1} \text{s}^{-1}$      |
| $\mu$               | collective entrainment rate in 1/s   |
| $\tilde{\mu}$       | collective entrainment rate per unit length in $\text{m}^{-1} \text{s}^{-1}$ |
| $\nu_{in}$          | inflow rate in beads/s   |
| $\nu_{out}$         | outflow rate in 1/s  |
| $\tilde{\nu}_{out}$ | outflow rate per unit length in $\text{m}^{-1} \text{s}^{-1}$                |

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| variable | meaning   |
|----------|---|
| $\omega$ | angular frequency   |
| $\phi_p$ | bed porosity in Eq. (14)  |
| $\psi$   | $\psi = 1 - \phi_p$ in Eq. (19)                                     |
| $\rho$   | fluid density in $\text{kg m}^{-3}$                                 |
| $\rho_p$ | particle density in $\text{kg m}^{-3}$                              |
| $\sigma$ | deposition rate in $1/\text{s}$                                     |
| $\tau_b$ | bottom shear stress in Pa   |
| $\tau_c$ | critical bottom shear stress (threshold for incipient motion) in Pa |
| $\theta$ | mean bed slope  |
| $\zeta$  | random number, see Eq. (44)   |

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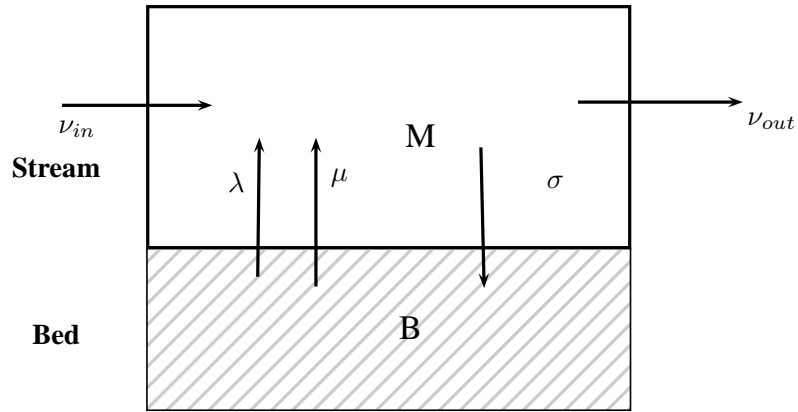


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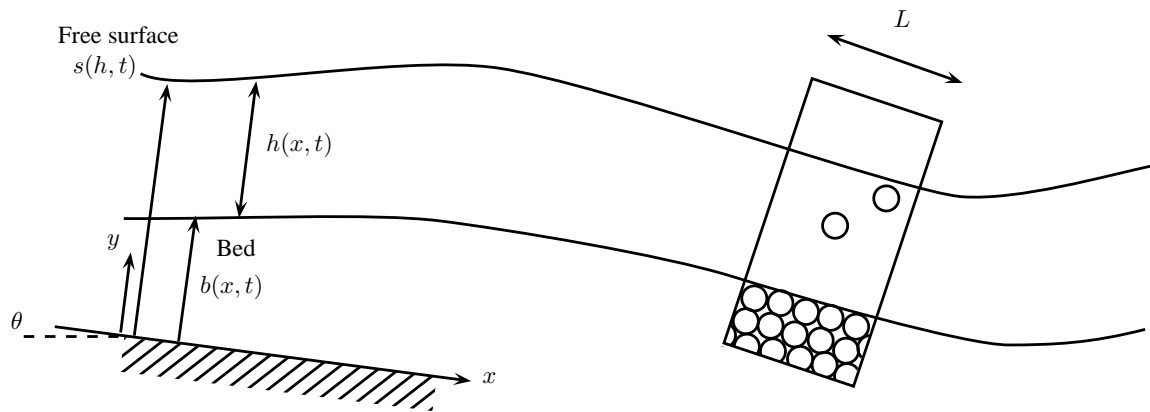
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**Figure 1.** The number of moving particles observed within the window varies with time depending on the number of particles entering/leaving the window or being entrained/deposited from/on the stationary bed.



**Figure 2.** Definition sketch of the model. The reference level is the  $x$ -axis.

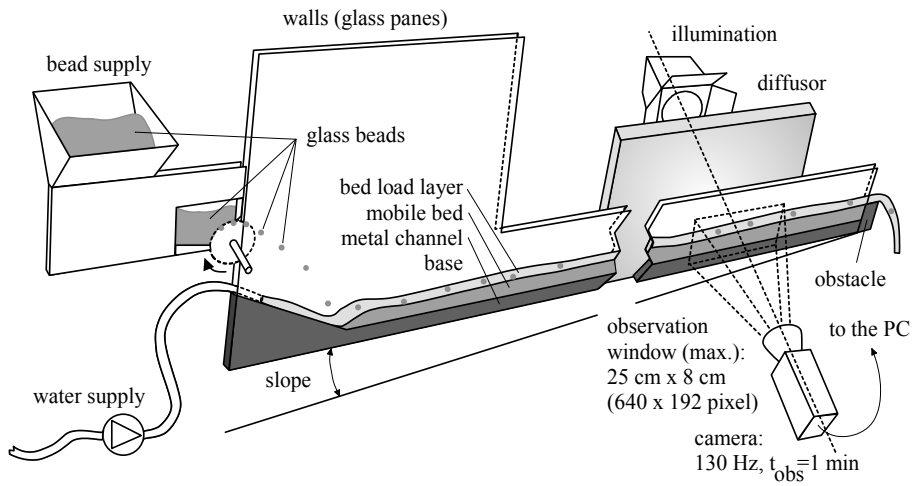


Figure 3. Sketch of the experimental setup.

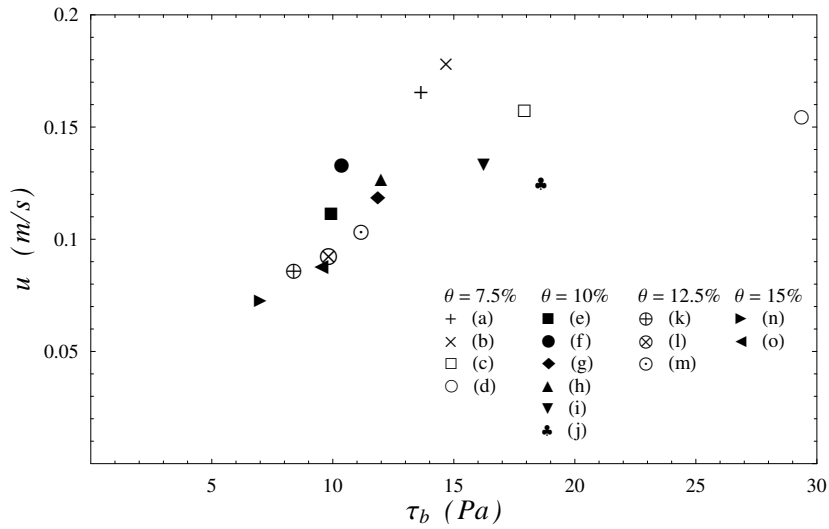
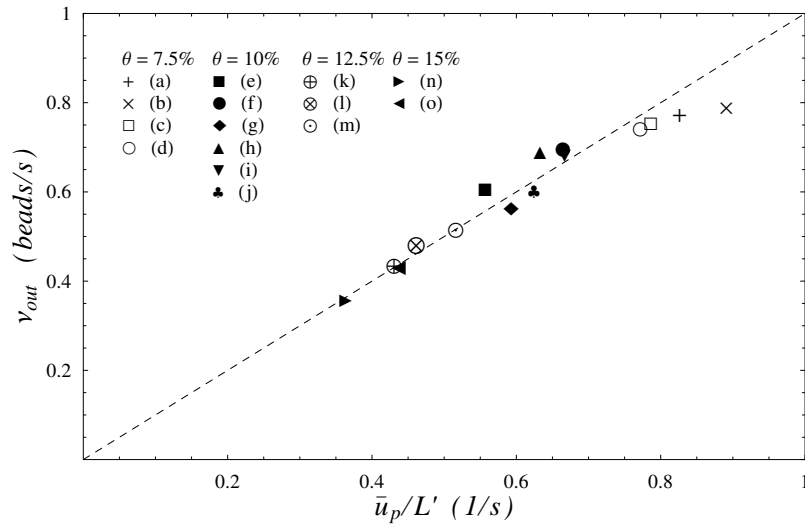
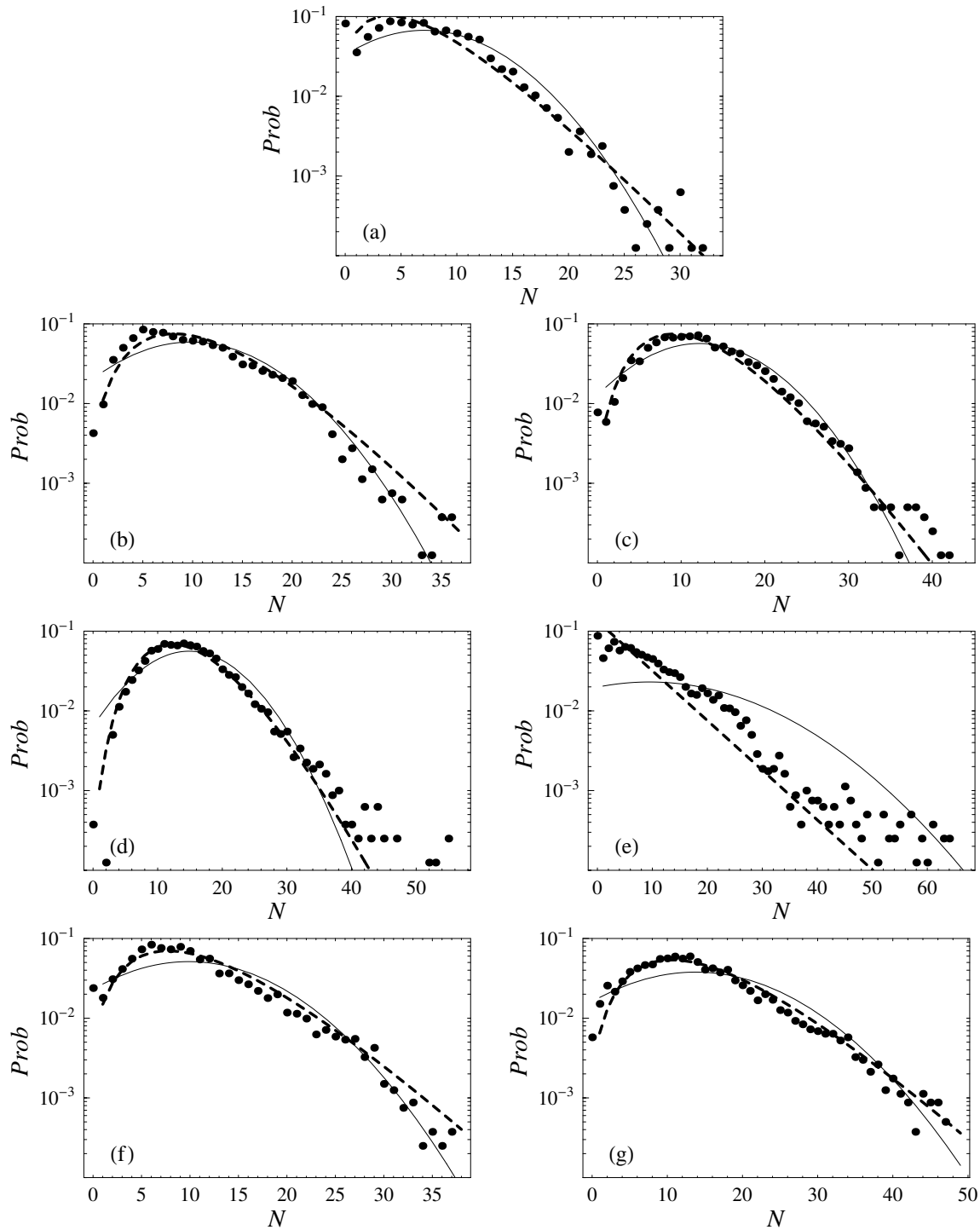


Figure 4. Relation between the bottom shear stress  $\tau_b$  and the mean particle velocity

$\bar{u}_p$ .

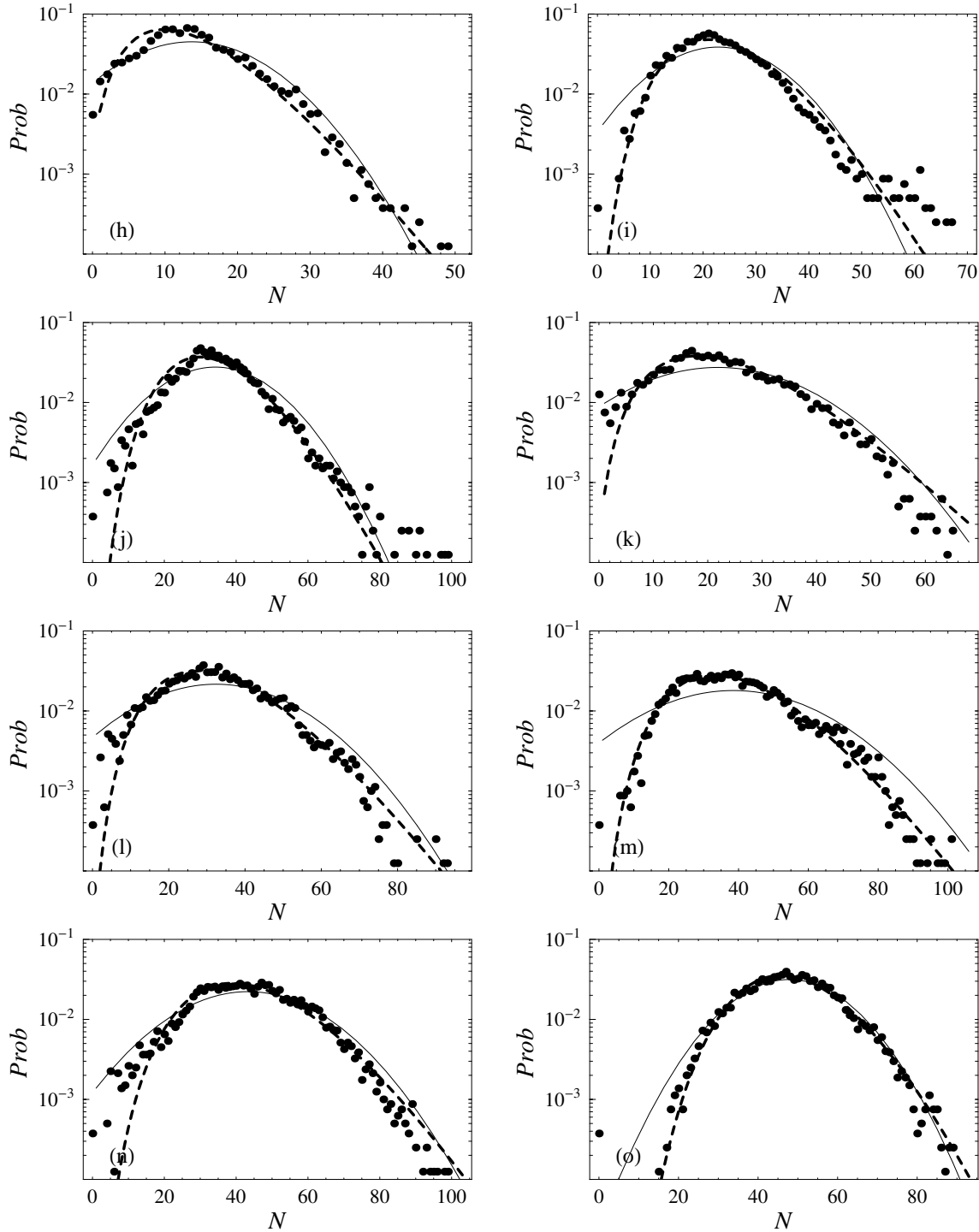


**Figure 5.** Relation between the measured  $\nu_{out}$  values and the ratio  $\bar{u}_p/L'$  with  $L' = 20$  cm.

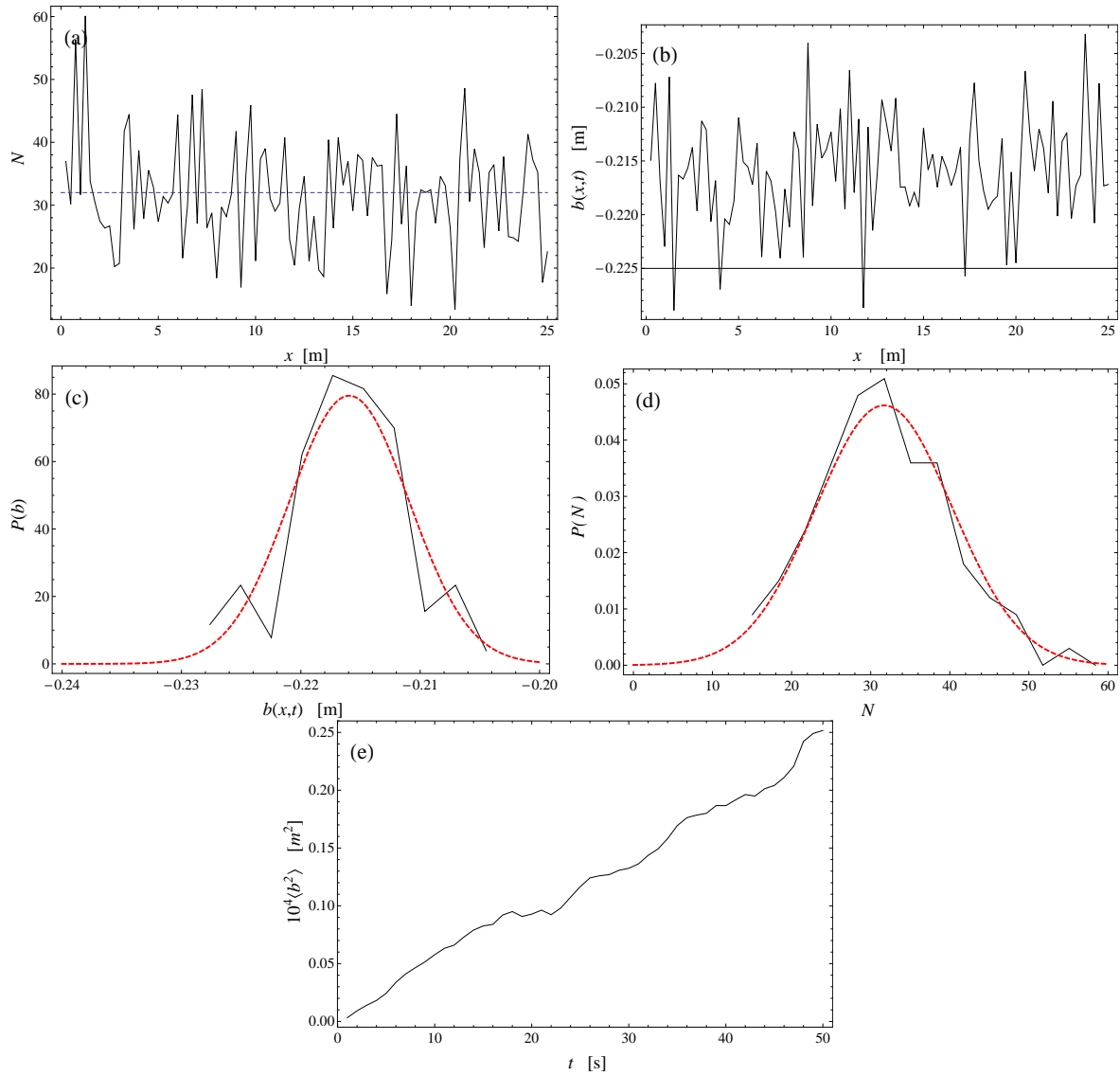


**Figure 6.** Empirical probability density of the total number of moving beads  $N$  (black dots). The dashed line is the probability density function of the negative binomial distribution. Experiments (a–i). The solid line represents the Gaussian approximation (34).

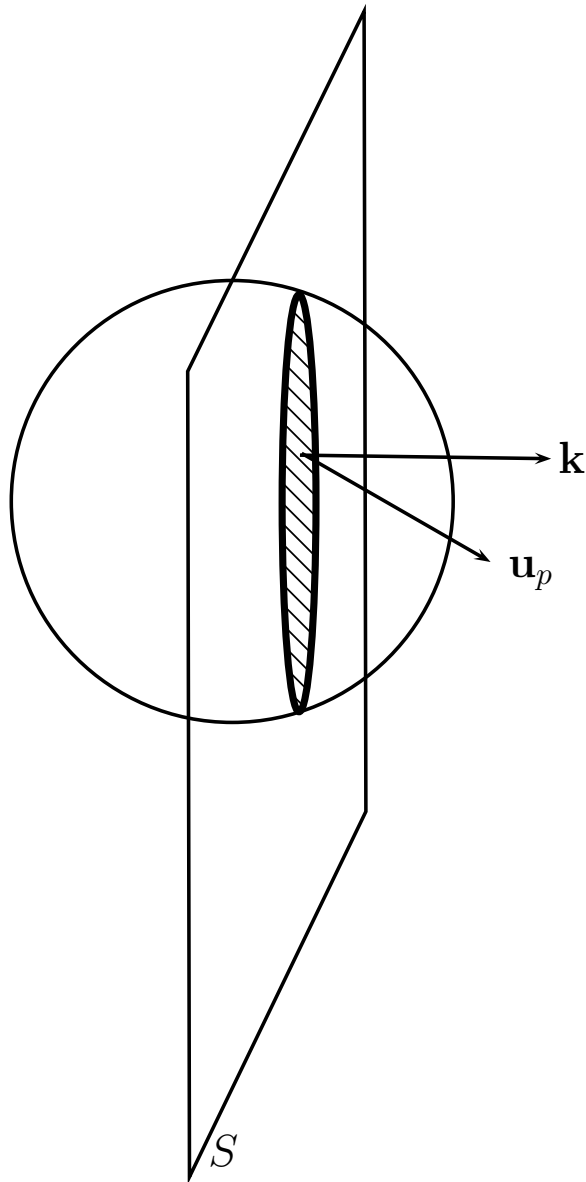




**Figure 6.** (Continued). Empirical probability distribution of the total number of moving beads  $N$  (black dots). The dashed line is the probability density function of the negative binomial distribution. Experiments (j–o) The solid line represents the Gaussian approximation (34).



**Figure 7.** Numerical results  $\Delta x = 0.25$  m,  $\Delta t = 10^{-4}$  s; the exchanges rates are grossly estimated from our experiments (see Appendix A):  $\lambda(\tau_b) = 2(\tau_b - 7)$  [beads/s],  $\mu(\tau_b) = 4$  [1/s],  $\sigma(\tau_b) = 5$  [1/s],  $\nu_{in}(\tau_b) = 5 + \lambda(\tau_b)/3$  [beads/s], and  $\nu_{out}(\tau_b) = \tau_b/20$  [1/s]. The particle radius is  $a = 3$  mm. (a) Variation of  $N$  along the flume at time  $t = 50$  s. (b) Bed-elevation profile at time  $t = 50$  s. (c) Empirical probability distribution of  $b$  as well as the fit by a normal distribution (dashed line) at  $t = 50$  s; the empirical distribution was built using the same data as those reported in subplot (b). (d) Empirical probability distribution of  $N$  as well as the fit by a normal distribution (dashed line) at  $t = 50$  s; the empirical distribution was built using the same data as those reported in subplot (a). (e)



**Figure 8.** Sketch showing the flux of particles through  $S$ .