Fluctuations of particle transport rates in graved-bed rivers or the quest for equilibrium?

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Outline

- Bed equilibrium?
- Microscopic analysis: birth-death markovian model
- Comparison with experiments

Reference: Ancey, C., and J. Heyman, A microstructural approach to bed load transport: mean behaviour and fluctuations of particle transport rates, *Journal of Fluid Mechanics*, **744**, 129-168, 2014.

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Morphodynamic models

Saint-Venant equations + Exner

$$\frac{\partial h}{\partial t} + \frac{\partial h\bar{v}}{\partial x} = 0, \quad (1)$$

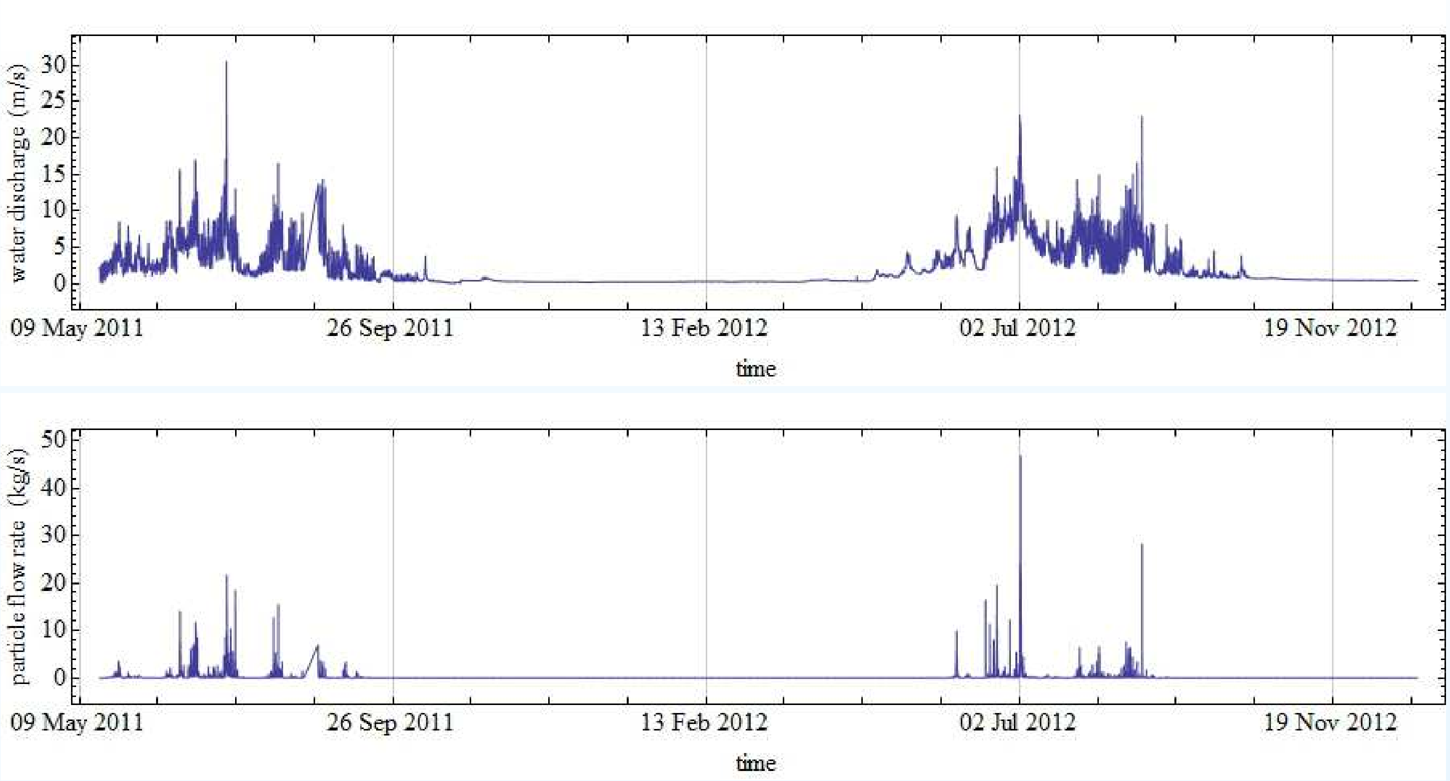
$$\frac{\partial h\bar{v}}{\partial t} + \frac{\partial h\bar{v}^2}{\partial x} + gh\cos\theta \frac{\partial h}{\partial x} = gh\sin\theta - \frac{\tau_b}{\varrho}, \quad (2)$$

$$(1-\zeta_b)\frac{\partial y_b}{\partial t} = -\frac{\partial \bar{q}_s}{\partial x} = D - E, \quad (3)$$

Nonlinear coupling between θ, y_b, $\bar{q}_s$, and τ_b

$\bar{q}_s$: particle flow rate per unit width [m^2/s], $\bar{v}$ depth-averaged velocity of water, h water depth, E and D entrainment and deposition rates, y_b bed elevation, ζ_b bed porosity, $\tan\theta = \partial_x y_b$ bed slope, ϱ water density, τ_b bottom shear stress

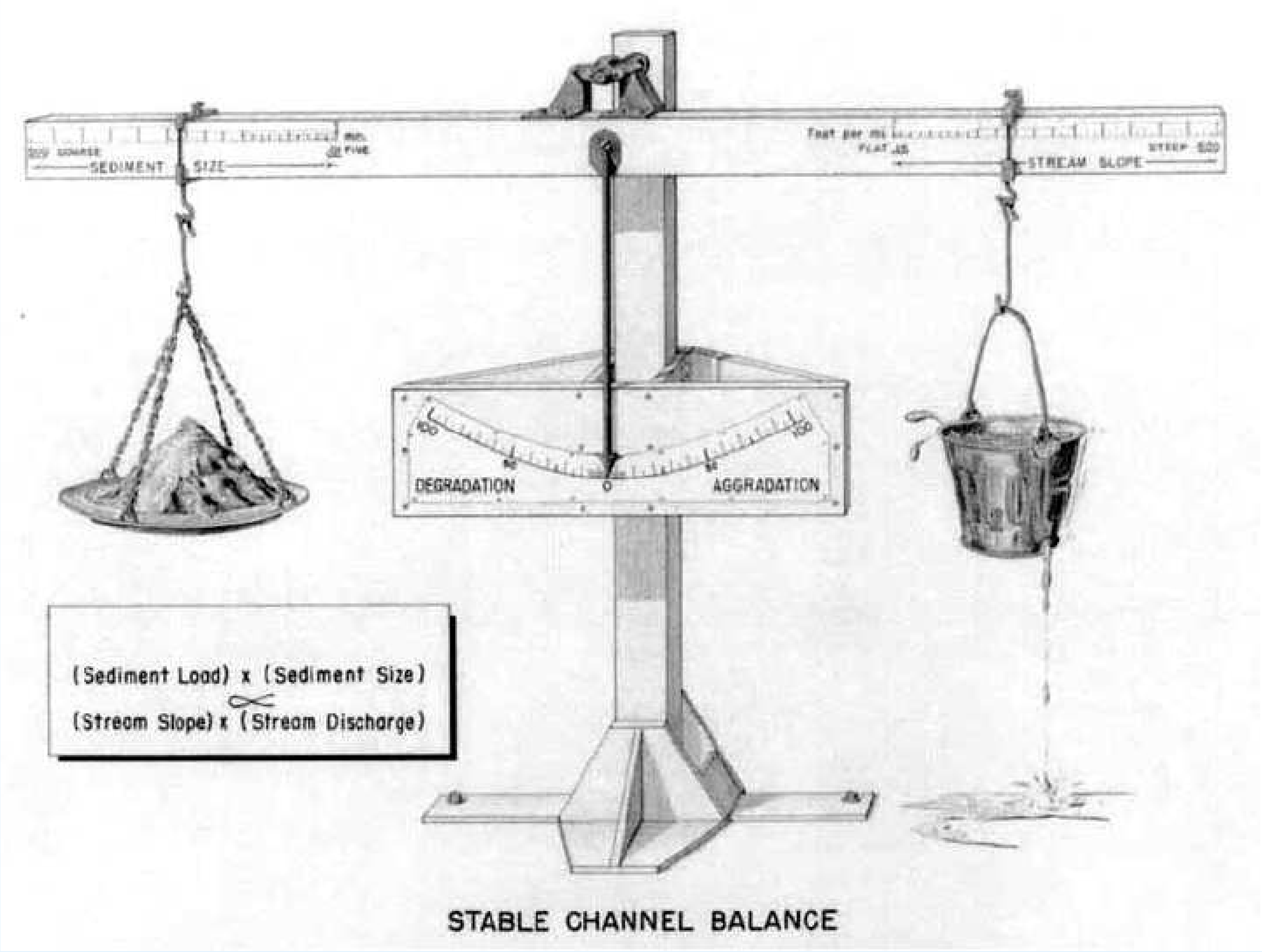
Fluctuations in the particle flow rates



Bedload transport measurements on the Navisence River (Valais, CH), bed slope 2 %

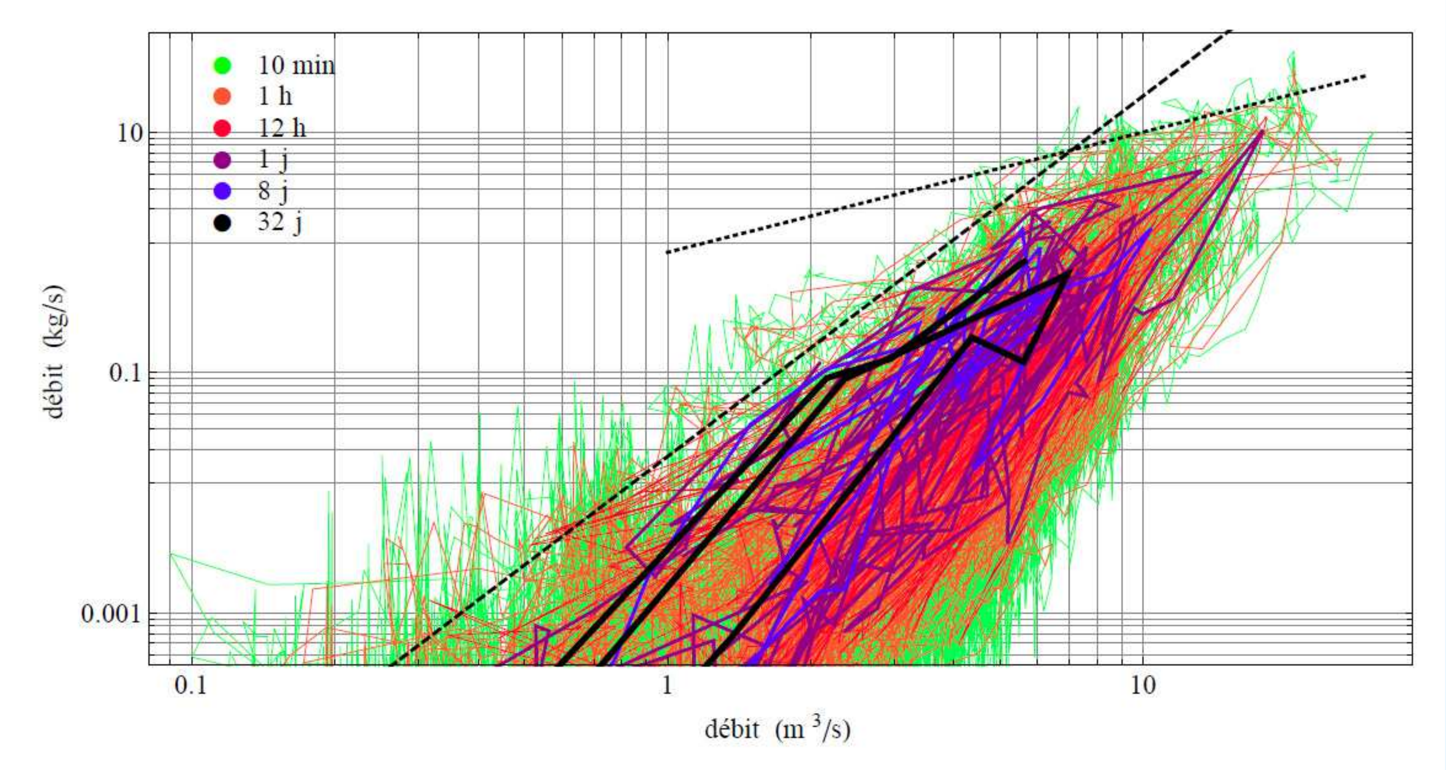
Courtoisy from Dr Eric Bardou

Traditional view: bed equilibrium



Lane, E.W., The importance of fluvial morphology in river hydraulic engineering, *American Society of Civil Engineers Proceedings*, 81, 1-17, 1955.

Equilibrium?

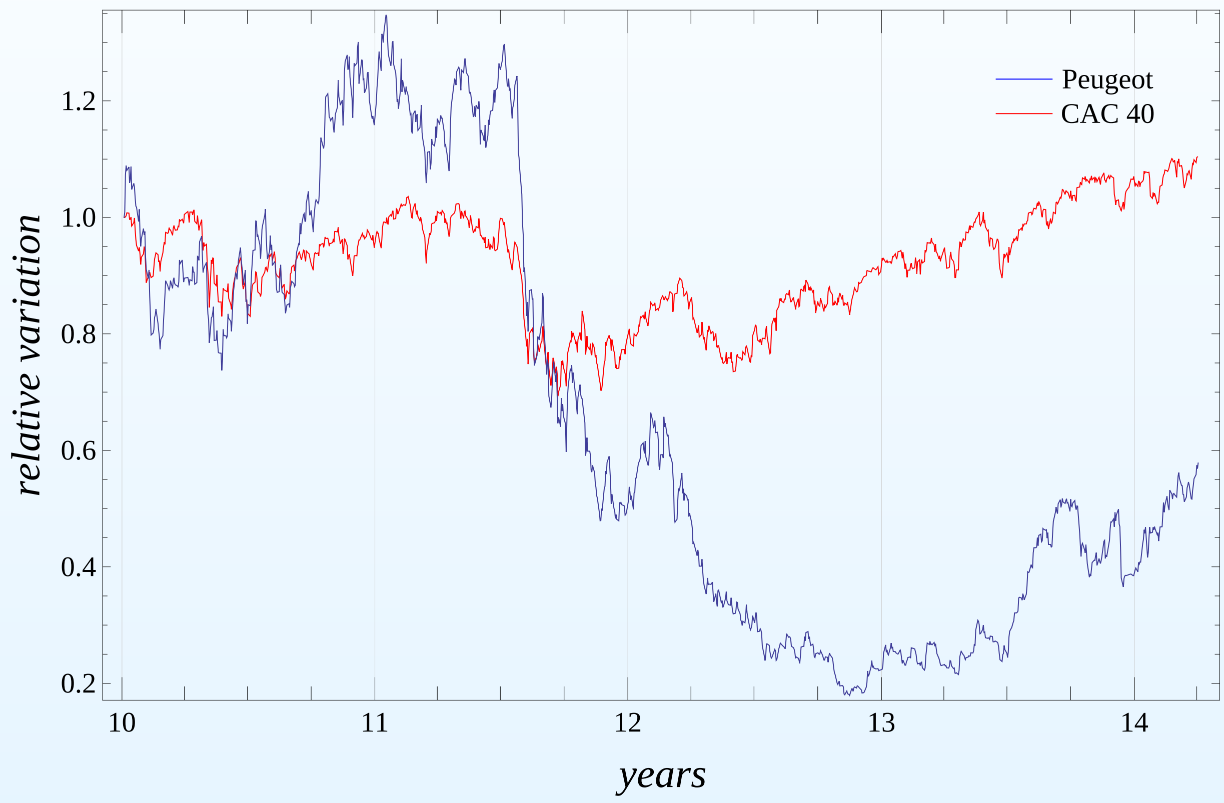


Relationship between Q_s and Q_l for different time spans

The dashed line represents the empirical trend $Q_s \propto Q_l^3$. The dotted line shows the linear trend $Q_s \propto Q_l$.

What do stock markets and sediment transport have in common?

Traditional view of economics: balance between offer and supply



If we were able to predict the mean trend, would we able to predict the evolution of a particular stock?

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Which Exner equation ?

- Jerolmack & Mohrig (WRR 2005)

$$(1-\zeta_b)\frac{\partial y_b}{\partial t} = -\frac{\partial \bar{q}_s}{\partial x} + \xi,$$

ξ white noise

- Furbish et al. (JGR 2012)

$$\bar{q}_s = \overline{u_s\gamma} - \frac{1}{2}\frac{\partial \overline{\kappa\gamma}}{\partial x}$$

κ particle diffusivity , γ particle activity (number of particles in motion per unit surface)

- Lajeunesse et al. (JGR 2010)

$$\ell_{sat}\frac{\partial \bar{q}_s}{\partial x} = q_{sat} - \bar{q}_s$$

ℓ_{sat} saturation length, q_{sat} saturation flow rate

Definition of the particle flow rate

The most natural definition: a flux of particles through a control surface

$$q_s = \int_S \mathbf{u}_p \cdot \mathbf{k} dS, \tag{4}$$

but this is difficult to apply to discrete objects. Probabilistic definition

$$\langle q_s \rangle = \int_S \int_{\mathbb{R}^2} P[\mathbf{u}_p \mid \mathbf{x},\, t]\; \mathbf{u}_p \cdot \mathbf{k}\; |d\mathbf{x}| d\mathbf{u}_p.$$

Other forms

- balance between deposition and erosion (Einstein) : $q_s = E\ell_s$;
- definition from the motion of tracers (Ferguson, Wong & Parker) : $q_s = U_p L_a$;
- definition from the particle trajectories (Furbish et al.) : $\bar{q}_s = \overline{u_s \gamma} - \frac{1}{2}\frac{\partial \overline{\kappa\gamma}}{\partial x}$;
- other definitions (Wiberg, Ballio & Nikora, etc.).

Our definition

Local particle flow (volume flow rate par unit width)

$$q_s(x,t;\mathcal{V};\bar{v},h) = N\frac{\varpi_p}{\Delta x}\mathcal{U}_n = \gamma\mathcal{U}_n = \frac{\varpi_p}{\Delta x}\sum_{i=1}^{N} u_{p,i},$$

or in terms of the number of particles per unit time

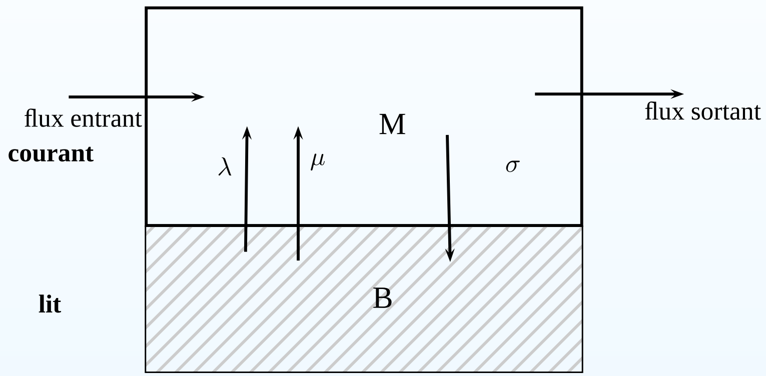
$$\dot{n} = \frac{N}{\Delta x}\mathcal{U}_n$$

- N number of moving particles ;
- $\gamma = N\varpi_p/\Delta x$: « particle activity » (Furbish) ;
- $\varpi_p = 4\pi d^3/(24B)$ particle volume per unit width (flow 1D) ;
- $\mathcal{U}_n = \sum_{i=1}^{N} u_{p,i}/N$ arithmetic average of the particle velocities.

Two key variables: mean particle velocity and number of moving particles

Model

Counting the number of particles in a control volume



Particle deposition

$$\mathrm{M} \xrightarrow{\sigma} \mathrm{B}$$

Entrainment of particles due to the stream

$$\mathrm{B} \xrightarrow{\lambda'} \mathrm{M}.$$

Collective entrainment (feedback loop)

$$\mathrm{M} + \mathrm{B} \xrightarrow{\mu} \mathrm{M} + \mathrm{M}.$$

Markovian formulation

For entrainment

$$P(n \to n{+}1, \Delta t) = \lambda' \Delta t{+}o(\Delta t), \quad P(n \to n{+}1, \Delta t) = \mu n \Delta t{+}o($$

For deposition

$$P(n \to n-1, \Delta t) = n\sigma\Delta t + o(\Delta t).$$

We end up with the Chapman-Kolmogorov for discrete variables

$$P(n, t+\Delta t) = \sum_{-\infty}^{+\infty} P(n+i, t) P(n+i \to n, \Delta t),$$

which leads to the master equation in the limit of $\Delta t \to 0$

$$\begin{aligned} \frac{\partial}{\partial t} P(n,t) &= (n+1)\sigma P(n+1,t) + \left(\lambda' + (n-1)\mu\right) P(n-1,t) \\ &- \left(\lambda' + n(\sigma+\mu)\right) P(n,t), \end{aligned}$$

How to solve the master equation?

The generating function

$$G(z,t) = \sum_{n=0}^{\infty} z^n P(n;t),$$

allows us to transform the master equation into a hyperbolic PDE

$$\frac{\partial}{\partial t} G(z,t) = L_z[G]$$

with the linear operator

$$L_z[G] = \lambda'(z-1) + \left\{\sigma + \mu z^2 - (\mu+\sigma) z\right\} \frac{\partial}{\partial z} G.$$

An equation that can be solved analytically...

Poisson transformation

The solution is the negative binomiale distribution, which can be seen as a Poisson distribution whose rate varies randomly

$$
\begin{aligned}
P_s(n) &= \int_a \frac{e^{-a}a^n}{n!}\mathrm{Ga}(a;\alpha,\beta)\mathrm{d}a \\
&= \mathrm{NegBin}(n, r_{nb}, p) \qquad (5)\\
&= \frac{\Gamma[r_{nb}+n]}{\Gamma[r_{nb}]}p^{r_{nb}}(1-p)^n \qquad (6)
\end{aligned}
$$

with $\alpha = r_{nb} = \lambda'/\mu$ and $\beta = 1/p - 1 = \mu/(\sigma - \mu)$

Poisson representation: a kind of Fourier transform

$$P(n,t) = \int_{\mathcal{C}} \frac{e^{-a}a^n}{n!} f(a,t)\mathrm{d}a,$$

in probability spaces that makes it possible to work with continuous random variables

Poisson representation

We show that we can transform

$$\frac{\partial G}{\partial t} = L_z[G] \text{ with } G(z,t) = \{K(z,a), f(a,t)\},$$

(with $K(z,a) = \exp[a(z-1)]$ the Laplace kernel), into a second order parabolic PDE

$$\frac{\partial f}{\partial t} = \mu \frac{\partial^2 a f}{\partial a^2} - \frac{\partial}{\partial a}[(\lambda' - a(\sigma - \mu))f].$$

Strengths:

- looks like the Fokker-Planck equation;
- well-known from the mathematical standpoint (Feller process, 1950)...
- used in economics (Cox-Irgensoll-Ross model of interest rates, 1985);
- can be generalized to multidimensional problems (array of adjacent cells).

Velocity luctuations

Analogy with Brownian motion

$$\frac{\mathrm{d}X}{\mathrm{d}t} = U,$$
$$t_r\frac{\mathrm{d}U}{\mathrm{d}t} = -(U-\bar{u}_s)+\sqrt{2D_u}\xi(t),$$

where X denotes the abscissa of the center of mass, U the velocity, t_r the relaxation time, D_u particle diffusivity, $\xi(t)$ white noise. This system is equivalent to the Fokker-Planck equation

$$\frac{\partial \hat{P}}{\partial t} = -\frac{\partial}{\partial x}(u\hat{P}) + \frac{\partial}{\partial u}\left(\hat{P}\frac{u-\bar{u}_s}{t_r}\right) + \frac{1}{t_r^2}\frac{\partial^2}{\partial u^2}(D_u\hat{P}).$$

with $\hat{P}(x,u,t)$ the Lagrangian probability of observing (x,u) at time t. At equilibrium, we get

$$P_u^{eq}(u) = \sqrt{\frac{2t_r}{\pi D_u}}\frac{1}{1+\mathrm{erf}(\bar{u}_s\sqrt{t_r}/\sqrt{2D_u})}\exp\left(-\frac{t_r(u-\bar{u}_s)^2}{2D_u}\right),$$

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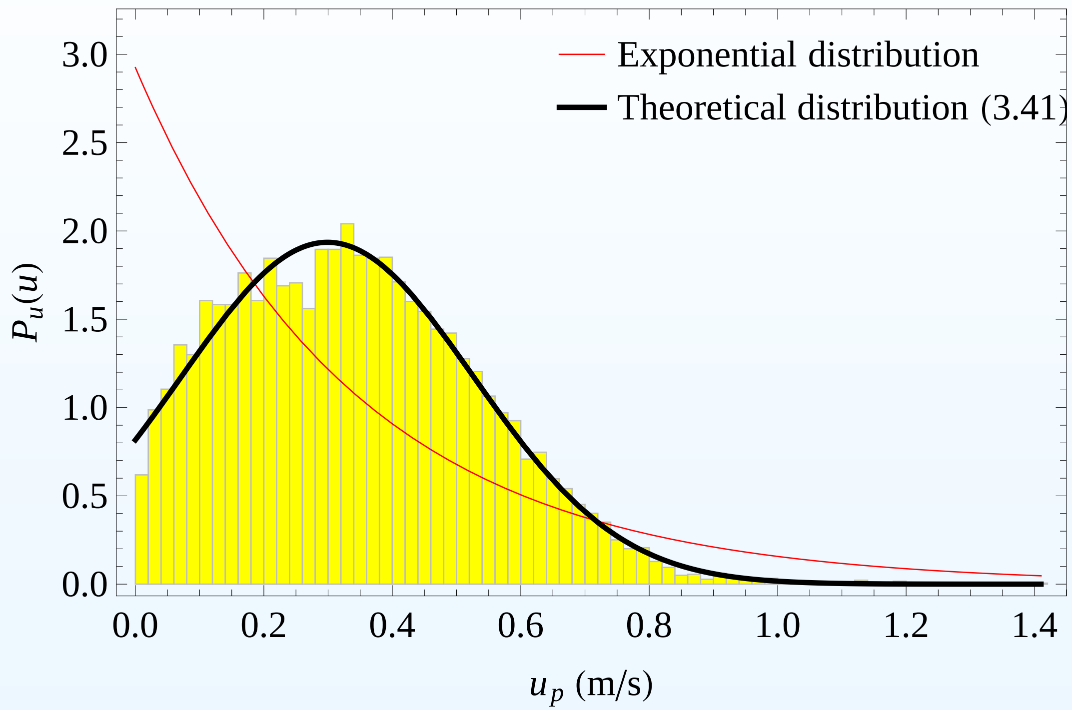

Probability distribution for the particle velocity. The histogram represents the empirical probability density function of u_p. The thick (black) solid line is the theoretical distribution with $\bar{u}_s = 29.9$ cm s^{-1} and $\zeta = 5.7$. The thin line shows the exponential probability distribution $P_u(u) = \mathrm{e}^{-u/\bar{u}_s}/\bar{u}_s$, still with $\bar{u}_s = 29.9$ cm s^{-1}. Computations on 755 trajectories, flume inclination 1.6°, supercritical flow (Froude number $Fr = 2.1$) and turbulent ($Re \approx 18\,000$).

Calculation of the flow rate

We define the particle flow rate

$$\dot{n}(t;\mathcal{V}) = \frac{1}{\Delta x}\sum_{i=1}^{N(t)} U_{p,i},$$

which leads to

$$P_{\dot{n}}(\dot{n}) = P_s(0)\delta(\dot{n}) + \frac{\zeta\Delta x}{\bar{u}_s}\sqrt{\frac{2}{\pi}}\sum_{k=1}^{\infty} P_s(k)\frac{\exp\left[-\zeta^2\frac{(\dot{n}\Delta x - k\bar{u}_s)^2}{2k\bar{u}_s^2}\right]}{\sqrt{k}(1+\mathrm{erf}(\sqrt{k}\zeta/\sqrt{2}))},$$

The following approximation holds for $\zeta > 2$

$$\langle\dot{n}(t;\mathcal{V})\rangle \approx \frac{1}{\Delta x}\langle N\rangle\langle u\rangle_{eq.} = \frac{\bar{u}_s}{\Delta x}\frac{\lambda'}{\sigma-\mu}F_{eq.}(\zeta),$$

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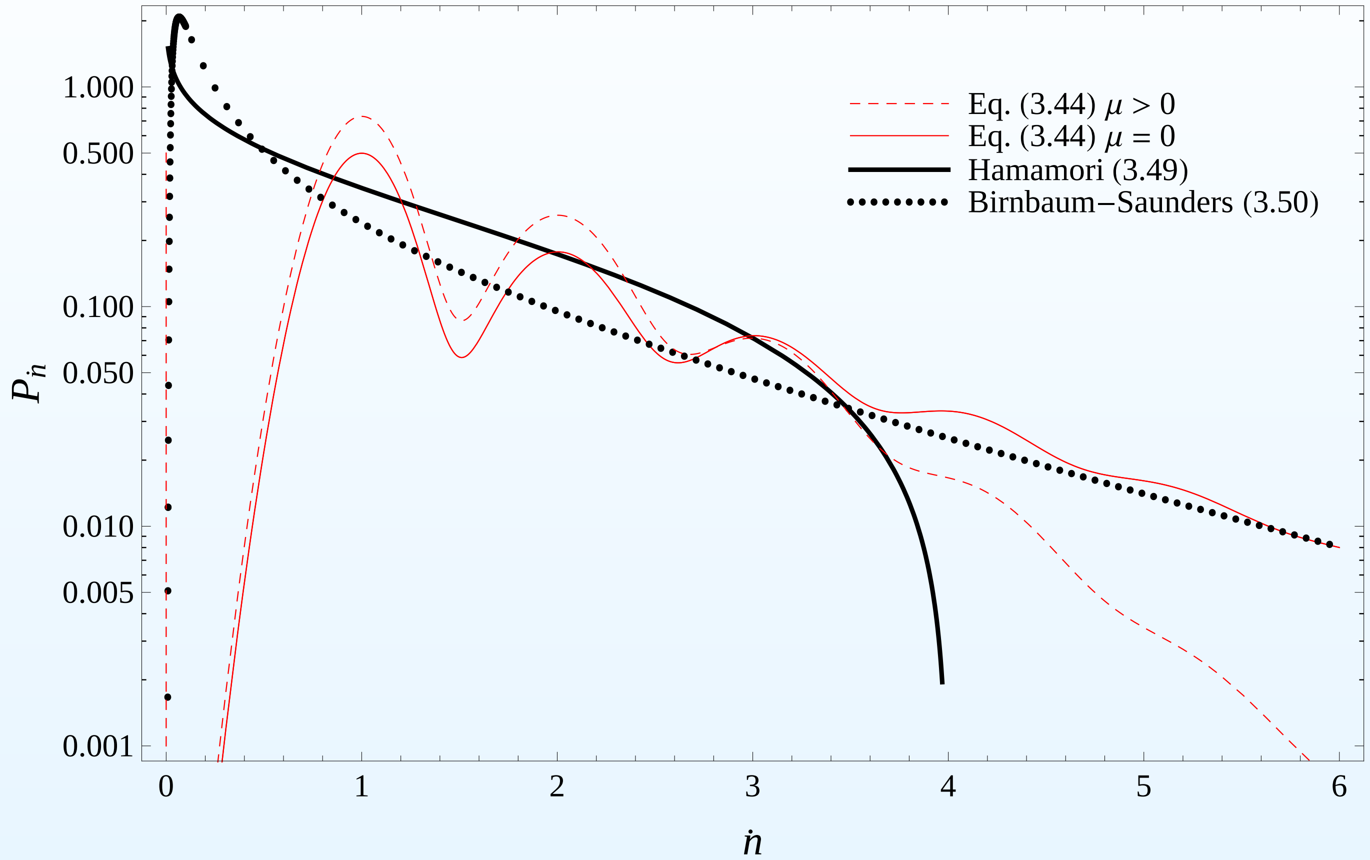
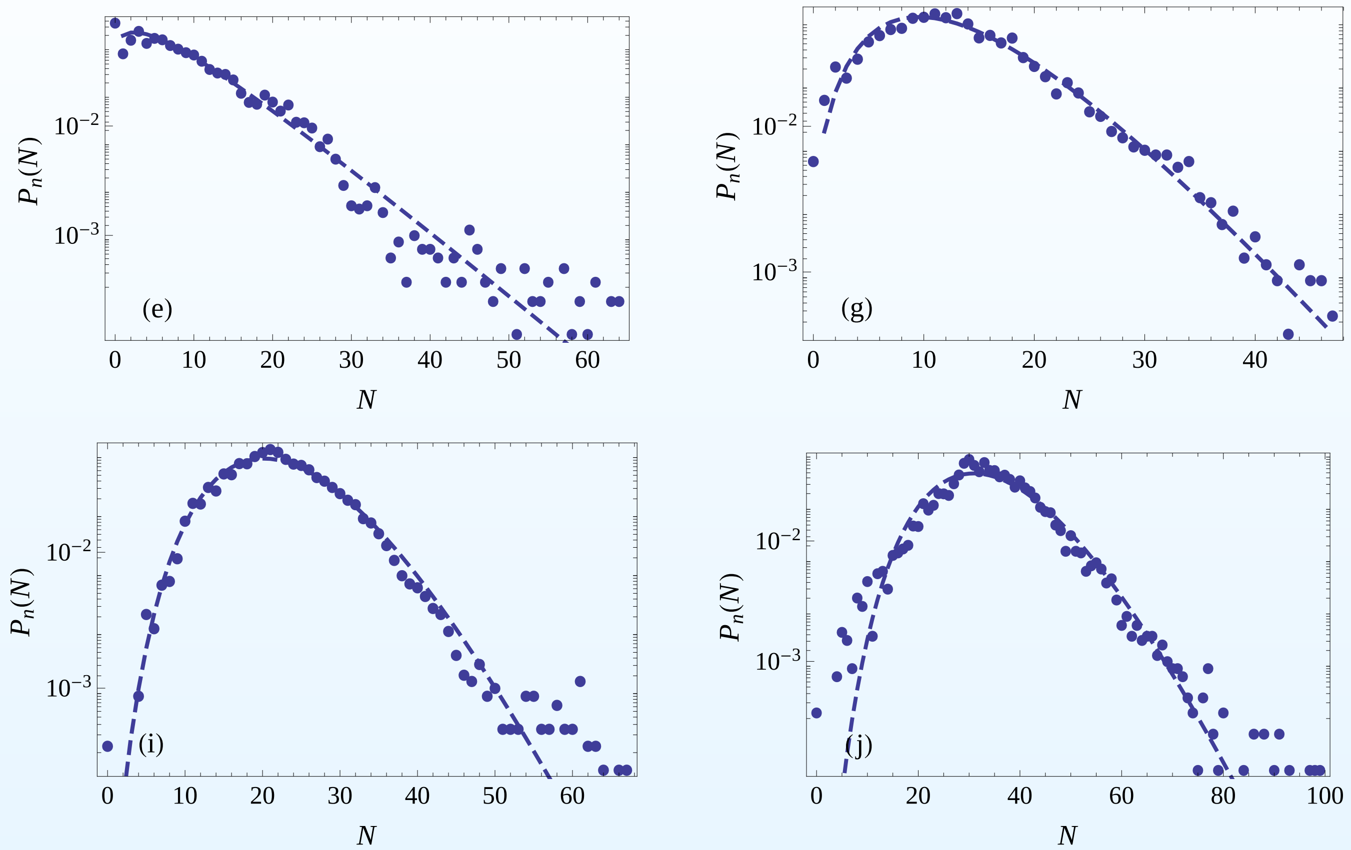
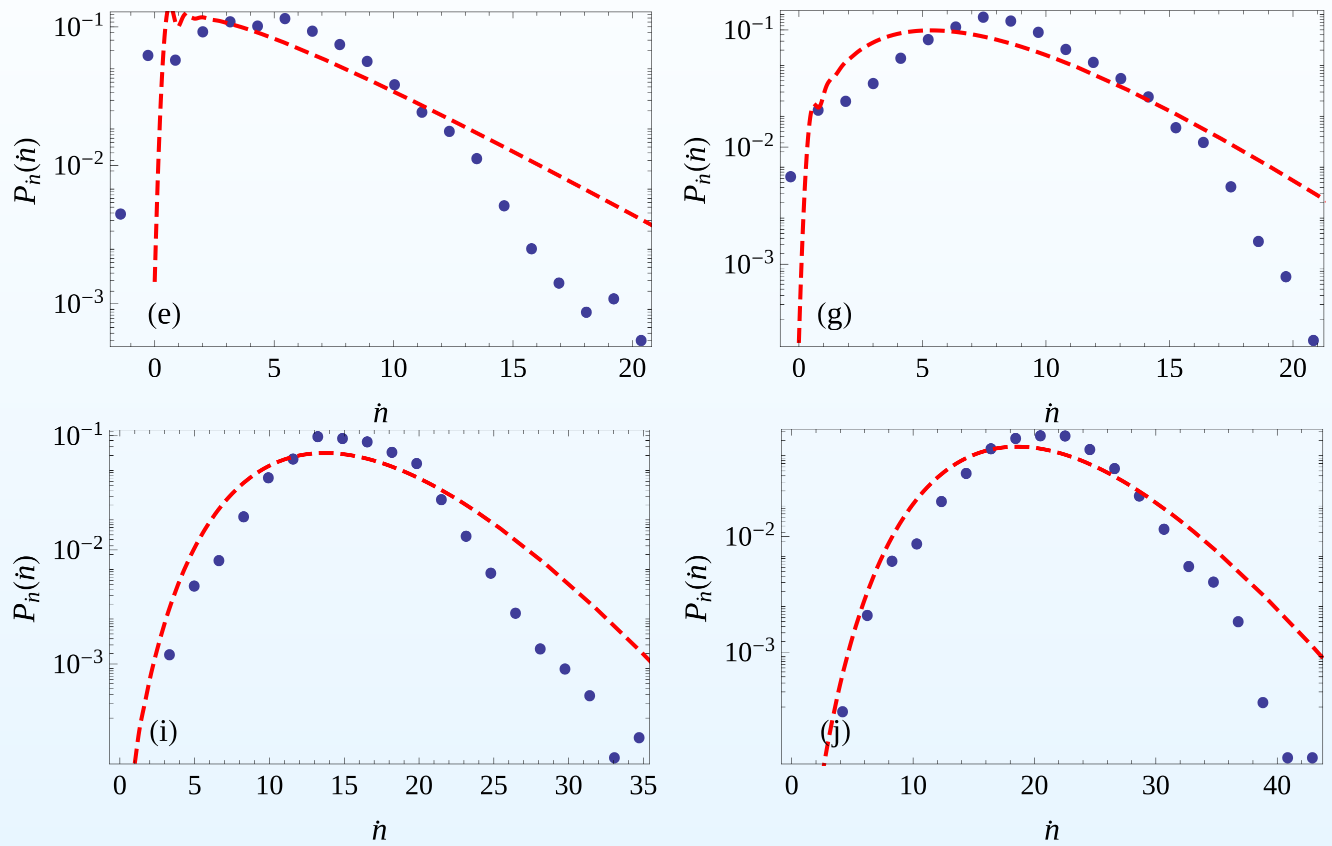


Comparison of the probability density function $P_{\dot{n}}(\dot{n})$ in a loglinear plot: Hamamori's equation and Birnbaum-Saunders (Turowski WRR 2010)

Comparison with experiments



Probability density functions for the number of moving particles in experiments. Water velocity: (e) 41 cm s^{-1}, (g) 44 cm s^{-1}, (i) 48 cm s^{-1}, and (j) 53 cm s^{-1}.



Probability density functions for the particle flux in experiments $P_{\dot{n}}(\dot{n})$ and comparison with experimental data.

Strengths and weaknesses

Some limitations of the approach in its current formulation

- strong coupling: the solid phase is subordinate to the fluid phase
- one-dimensional flow
- identical spherical particles: no sorting, no armouring, etc.
- the model holds for a limited range of $\zeta = \bar{u}_s/\sqrt{D_u/t_r} : \zeta > 2$ for the generalization to adjacent cells to work.

Summary

Highlights

- If we define the bulk particle flow rate as

$$\mathcal{Q}(x,t) = \langle\gamma\rangle\bar{u}_s - \frac{\partial}{\partial x}(D_u\langle\gamma\rangle),$$

  then we can deduce the Exner equation from our microstructural analysis of the stochastic motion of particles;
- importance of collective entrainment: feedback loop that exacerbates the q_s fluctuations;
- modelling at the particle level: numerical simulations can be performed using the Fokker-Planck equation obtained.