Chapter 1: Concept of similarity

Similarity and Transport Phenomena in Fluid Dynamics

Christophe Aney
Chapter 1: Concept of similarity

- A few definitions
  - Geometrical similarity
  - Fractal similarity
  - Physical Similarity
- Scaling law
- Complete similarity
- Incomplete similarity
- Historical background
Geometrical similarity

Triangles are geometrically *similar* if

\[ \lambda = \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}, \]

with \( \lambda \) the *similarity ratio* (also called the *scale factor* or *scale*). This is an example of *isomorphic transformation*. 
Geometrical similarity

Generalisation: an affine transformation conserves length ratios, with axis-dependent ratios

$$\lambda_x = \frac{a'}{a} \text{ et } \lambda_y = \frac{b'}{b},$$

with $\lambda_x$ et $\lambda_y$ the ratios in the horizontal and vertical direction.
Geometrical similarity

Transformations that leave an object invariant

• Discrete transformation, e.g., rotation by $\pi/2$ for a square

• Continuous transformation, e.g., rotation by an angle $\theta$ for a circle, translation by $\epsilon$ for a straight line

Infinitesimal continuous transformations are key to constructing exact solutions.
Some continuous transformations $\Gamma(\theta)$ are one-dimensional Lie groups

- $\Gamma(0)$ is identity $Id$
- $\Gamma(\theta_1) \circ \Gamma(\theta_2) = \Gamma(\theta_1 + \theta_2)$ (rule of closure)
- the inverse of $\Gamma(\theta)$ is $\Gamma(-\theta)$, and $\Gamma(\theta) \circ \Gamma(-\theta) = Id$
- $(\Gamma(\theta_1) \circ \Gamma(\theta_2)) \circ \Gamma(\theta_3) = \Gamma(\theta_1) \circ (\Gamma(\theta_2) \circ \Gamma(\theta_3))$ (associativity)

Additional property: each image may be represented as a Taylor series in $\theta$ (for $\theta \to 0$)

$$\hat{x} = x + \theta \partial_x \Gamma + O(\theta^2).$$
Exercise 1

Can you find symmetries for this equation?

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

Solve this equation and comment on.
Geometrical similarity: invariant

When applying an affine transformation,

- certain quantities are conserved. They are called *invariants*. For instance, taking the ratio between the surface $S$ and the product of half axes:

$$s = \frac{S}{ab} = \frac{S'}{a'b'} = \pi.$$  

- other quantities are not conserved. This is the case of the perimeter

$$P = 4 \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta$$
Scale invariance

An angle is scale invariant

\[ \alpha = \frac{s}{r} = \frac{s'}{r'} \]

Scale invariant, but not affine invariant.
Why are certain quantities conserved, whereas others are not? 

A scaling law describes the proportionality relation between a quantity and the scale(s) of the problem:

- perimeter $P \propto \ell$,
- area $S \propto \ell^2$,
- volume $V \propto \ell^3$,

with $\ell$ a length scale of the object.
Scale invariance

Log-log plot of the energy spectrum in the time domain in helium flow between rotating cylinders

\[ E(f) = f^{-5/3} \]

(Frisch, U., Turbulence, Cambridge University Press, Cambridge, 1995; p. 66)
Log-log plot of the basal metabolic rate of mammals and birds (in kcal/day) with mass ($M$ in kg)

$$\text{BMR} \propto M^{-3/4}$$

Scale invariance

Zipf’s law applied to agglomeration size

\[ \text{rank} \propto \text{size}^{-0.76} \]

Brownian motion: the aspect looks the same in the two enlargements

(Frisch, U., Turbulence, Cambridge University Press, Cambridge, 1995; p. 66)
Fractal similarity
Fractal similarity

Rivers forming tree-like figures in the desert of Baja California, Mexico (© Adriana Franco)
Invariance
Exercises 2 and 3

2. Consider a rectangular triangle whose sides are $a$, $b$, $c$ ($a$ is the hypothenuse). Plot the altitude perpendicular to the hypothenuse (In geometry, an altitude of a triangle is a line segment through a vertex and perpendicular to a line containing the base—the opposite side of the triangle). What can you say about of the two triangles split by the altitude? Can you give a proof of Pythagoras’ theorem?

3. Calculate the surface of an ellipse (Hint: make use of the affinity between a circle and an ellipse).
We calculate the drag force $F$ exerted by an incompressible Newtonian fluid on a spherical particles with diameter $2r$ and density $\varrho_p$.

There are 5 variables: (1) the (unknown) drag force, (2) fluid viscosity $\mu$, (3) fluid density $\varrho$, (4) particle radius $r$, and (5) the relative velocity $u = |\mathbf{u}_p - \mathbf{u}_f|$. The Vashy-Buckingham theorem states that we can build $5 - 3 = 2$ dimensionless numbers.
Physical similarity

The general dimensionless expression for the drag force read $\psi(Cd, \ Re) = 0$ or $Cd = \phi(Re)$:

$$Cd = \frac{F}{\frac{1}{2}\pi \rho r^2 u^2} = \phi(Re).$$

$Cd$ is the drag coefficient. Solving the Navier-Stokes equations in the limit $Re \ll 1$ yields:

$$Cd = \frac{F}{\frac{1}{2}\pi \rho r^2 u^2} = \phi(Re) = \frac{24}{Re} \quad \text{quand} \quad Re \to 0.$$

This is the Stokes law. In the limit $Re \gg 1$, experiments show that:

$$Cd = \frac{F}{\frac{1}{2}\pi \rho r^2 u^2} = \phi(Re) \approx 0, 4 - 0, 5 \quad \text{when} \quad Re \to \infty.$$
Variation of the drag coefficient with the particle Reynolds number: 

\[ C_d = \frac{F}{\frac{1}{2} \pi d r^2 u^2} \]

and 

\[ \text{Re} = \frac{2 \rho r u}{\mu} : \]
Exercise 4

4. Consider a Newtonian fluid at rest and bounded by an infinite upper boundary. The material is suddenly sheared by moving the boundary at constant velocity. This is the Stokes’ first problem. Write the Navier-Stokes equations. How can you simplify them? What are the boundary conditions? Solve the equation and comment on the results. (Hint: Use the Vaschy-Buckingham theorem and transform the NS equation into an ODE).
5. Consider a Newtonian fluid at rest and bounded by an infinite upper boundary. The material is suddenly experiencing a (constant) body force (the boundary does not move). Write the Navier-Stokes equations. How can you simplify them? What are the boundary conditions? Solve the equation and comment on the results. (Hint: make a change of variable, then transform the PDE into an ODE).
We are concerned with a water stream, whose variables are: \( \bar{u} \) [m/s], \( h \) [m], \( g \) [m/s\(^2\)], and \( \theta \) [-]. The gravitational acceleration \( g \) and \( \theta \) are put together so that there are only \( n = 3 \) variables. There are \( r = 2 \) fundamental units: m and s. Therefore, there is \( n - r = 1 \) dimensionless group. This is the Froude number \( Fr = \frac{\bar{u}}{\sqrt{gh \sin \theta}} \). The physics of water streams would entirely be embodied in the relation

\[
Fr = cst \Rightarrow \bar{u} \propto \sqrt{gh \sin \theta}.
\]

This is the Chézy law.
But what about bed roughness?

Let us introduce the roughness scale $k_s$ [m]. There are $n = 4$ variables and $r = 2$ units. So there are 2 dimensionless groups, e.g., $\Pi_1 = Fr = \overline{u}/\sqrt{gh \sin \theta}$ and $\Pi_2 = k_s/h$, and among them, there exists a relation in the form:

$$\Pi_1 = f(\Pi_2) \Rightarrow \overline{u} = f(\frac{k_s}{h}) \sqrt{gh \sin \theta}.$$ 

Since flow depth is usually much larger than $k_s$, we expect $k_s/h \rightarrow 0$ and $f(\frac{k_s}{h})$ should tend to a constant. If so, this is barely different from the Chézy law!
Another possibility is to assume that \( f \) behaves like a power function

\[
f(\zeta) = \alpha \zeta^n,
\]

with \( \zeta = \frac{k_s}{h} \), \( \alpha \) a dimensionless number, and \( n \) an exponent. We refer to this behaviour as \textit{incomplete similarity}. Under this assumption, we end up with

\[
\Pi_1 = \alpha \Pi_2^n \Rightarrow \bar{u} = \alpha k_s^n h^{1/2-n} \sqrt{g \sin \theta}.
\]

Taking \( n = -1/6 \) (from experimental observation) leads to the Manning-Strickler flow resistance equation. Therefore the Strickler coefficient \( K \) is linked with \( k_s \):

\[
K = \alpha \sqrt{g} k_s^{-1/6}.
\]
Historical background

- 6th century BC: Thales of Miletus
- ~1830 Galois' group theory
- 1870–1900: Sophus Lie and Felix Klein (Backlund, Noether, etc.)
- mid 20th century: Soviet physical school (Zeldovitch, Barenblatt, Sedov, etc.)
- mid 20th century: American mathematical school (Bluman, Anco, Olver)
- follow-on: renormalization group theory (Kadanoff, Wilson)