ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 2: Transport phenomena in fluid dynamics

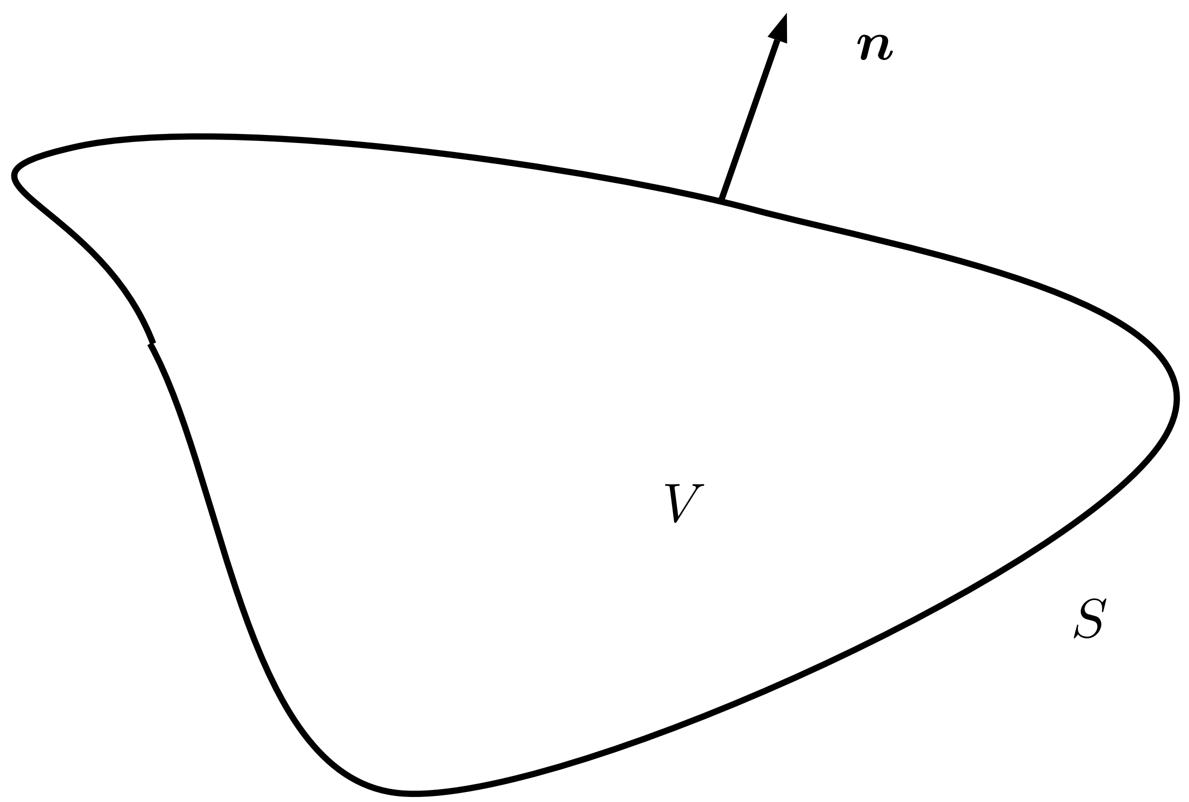
Similarity and Transport Phenomena in Fluid Dynamics

Christophe Ancey

Chapter 2: Transport phenomena in fluid dynamics

- Transport phenomena
  - Advection
  - Diffusion
  - Wave
- Boundaries
- Classification of PDEs

A short detour: conservation equations



Equations of conservation describe how the spatial and temporal variations of a function f are related. Let us consider a control volume V (and its bounding surface S), then

$$\int_V \frac{\partial f}{\partial t} \mathrm{d}V + \int_S f \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}S = \Phi$$

$\boldsymbol{n}$ normal oriented, $\boldsymbol{n}$ local velocity of the control surface, Φ rate of variation of f (if $\phi = 0$, then f is conserved).

A short detour: conservation equations

When f is continuous over V, then the Green-Ostrogradski theorem tell us that

$$\int_S f\boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}S = \int_V \nabla(f\boldsymbol{u})\mathrm{d}V$$

If this holds true for any volume V, then we can derive the local form of conservation

$$\frac{\partial f}{\partial t} + \nabla(f\boldsymbol{u}) = \phi$$

where ϕ is such that $\Phi = \int_V \phi\mathrm{d}V$.

Characteristic form of conservation equations

Quasi-linear first-order partial differential equations equations are linear in the differential terms. They can be put in the form:

$$P(x,\ y,\ u)\partial_x u + Q(x,\ y,\ u)\partial_y u = R(x,\ y,\ u).$$

The implicit solution can be written as $\psi(x,\ y,\ u(x,\ y)) = c$ (with c a constant). ψ is a first integral of the vector field $(P,\ Q,\ R)$. We have:

$$\partial_x \psi(x,\ y,\ u(x,\ y)) = 0 = \psi_x + \psi_u u_x,$$
$$\partial_y \psi(x,\ y,\ u(x,\ y)) = 0 = \psi_y + \psi_u u_y.$$

Characteristic form of conservation equations

We deduce that $u_x = -\psi_x/\psi_u$ et $u_y = -\psi_y/\psi_u$, and thus:

$$P\psi_x + Q\psi_y + R\psi_u = 0,$$

or in a vector form

$$(P, Q, R) \cdot \nabla\psi = 0.$$

Geometrical interpretation: At point M, the normal to the solution surface is normal to the vector field $(P,\ Q,\ R)$. If point O $(x,\ y,\ u)$ and neighbour point O' $(x + \mathrm{d}x,\ y + \mathrm{d}y,\ u + \mathrm{d}u)$ belong to the solution surface, then the vector $\mathbf{OO}'$ $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}u)$ is normal to $\nabla\psi$: $\psi_x dx + \psi_y dy + \psi_u du = 0$. We then deduce the characteristic form.

Characteristic form of conservation equations

The *characteristic equations* is

$$\boxed{\frac{\mathrm{d}x}{P(x,\ y,\ u)} = \frac{\mathrm{d}y}{Q(x,\ y,\ u)} = \frac{\mathrm{d}u}{R(x,\ y,\ u)}}$$

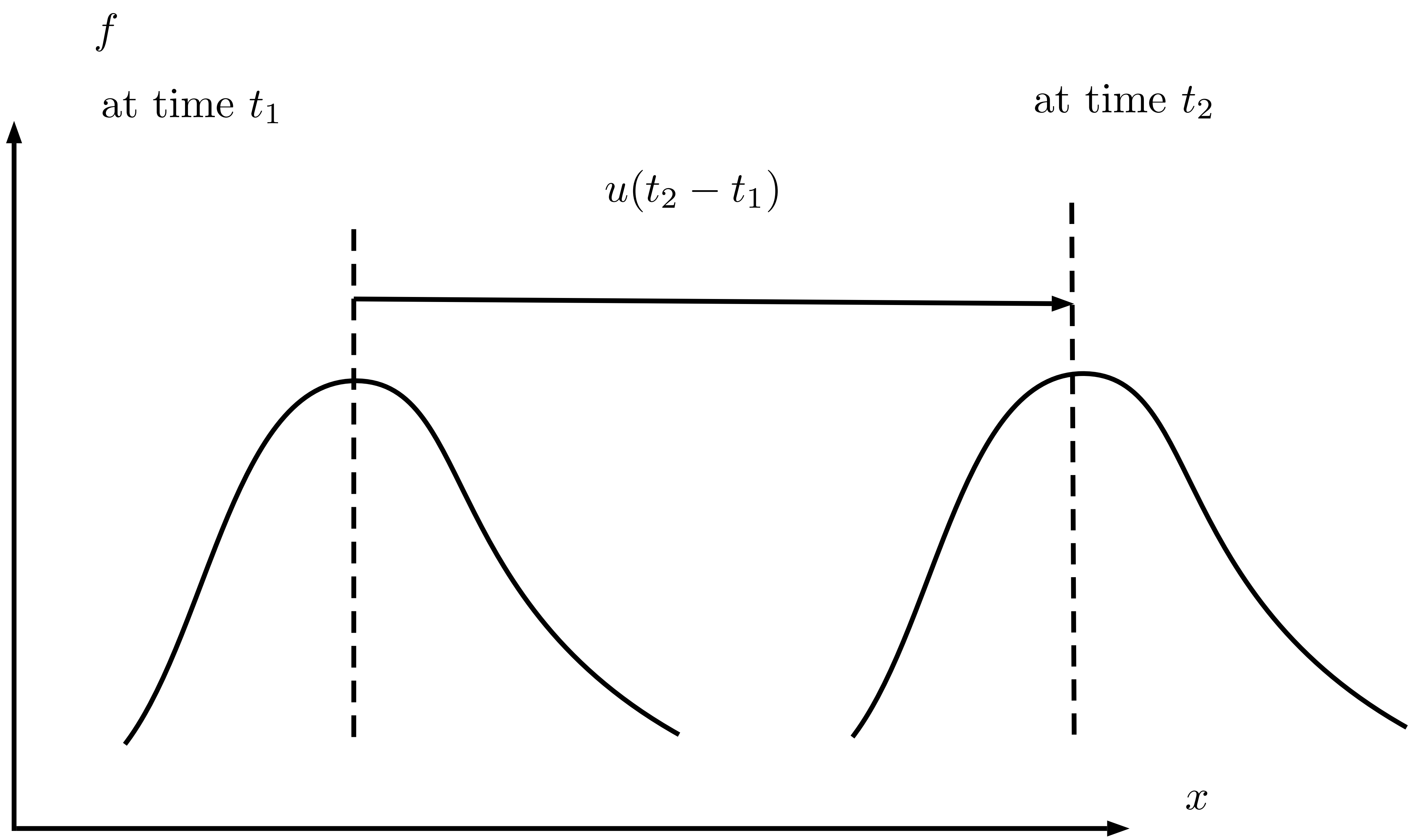
Each pair of equations defines a curve in the space $(x,\ y,\ u)$. These curves define a two-parameter family (there are 3 equations, so 3 invariants but only 2 are independent): for example, if p is a *first integral* of the first pair of equations, an integral surface of the first pair is given by an equation of the form $p(x,\ y,\ u) = a$, with a a constant. Similarly for the second pair: $q(x,\ y,\ u) = b$. The functional relation $F(a,\ b) = 0$ defines the integral curve.

Exercise 1

1. Solve equation

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u^2.$$

Advection



The simplest convection equation is the following one

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0,$$

Linear advection with no source term.

Advection

The characteristic equation associated to this PDE is

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}f}{0}.$$

As u is assumed to be constant, this means that the solution of the characteristic equation is $x - ut = const$ and any function $F(x - ut)$ whose argument is $x - ut$ is a solution. A feature of this solution is that the original form $F(x)$ (at $t = 0$) is conserved in the course of movement: it is simply translated by ut.

Advection: methods of characteristics

Let us consider the solution $F(x,t)$ to the linear advection equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

and a continuous path $X(t)$. The time derivative of F along X is

$$\frac{\mathrm{d}F}{\mathrm{d}t}(X(t),t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{\mathrm{d}X}{\mathrm{d}t}$$

If we select X such that $X' = u$ then

$$\frac{\mathrm{d}F}{\mathrm{d}t}(X(t),t) = 0$$

This means that F is constant along the path X. This shows the equivalence

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0 \Leftrightarrow \frac{\mathrm{d}f}{\mathrm{d}t} = 0 \text{ along } x(t) \text{ such that } \frac{\mathrm{d}x}{\mathrm{d}t} = u$$

Diffusion

An example of one-dimensional diffusion equations is the heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

with α thermal diffusion, $T(x,\ t)$ temperature, x abscissa in the bar direction. This is a second-order linear equation, which describes heat diffusion along the bar.

Example: spread of a heat source. Conservation of thermal energy E imposes

$$\int_{-\infty}^{\infty} T(x,\ t)\mathrm{d}x = V = \frac{E}{cS}.$$

with c heat capacity

Diffusion: Dimensional analysis and similarity solution

There are $n = 5$ variables: T, x, t, α, and V; the other variables (E, c, and S) are introduced through V.

The dimensional matrix is the following

	T	x	t	α	V
homogeneous to	K	m	s	m^2/s	m$\cdot$ K
power decomposition :					
power of m	0	1	0	2	1
power of s	0	0	1	-1	0
power of K	1	0	0	0	1

Diffusion: Dimensional analysis and similarity solution

This is a 3×5 matrix of rank 3 (the fourth column is obtained by linear combination of columns 2 and 3, column 5 is the sum of columns 1 and 2). We can therefore form $k = n - r = 2$ dimensionless numbers. Let us pose

$$\Pi_1 = x\alpha^a t^b V^c \text{ et } \Pi_2 = T\alpha^{a'} t^{b'} V^{c'}.$$

To get $[\Pi_1] = 0$, we must have

$$[\mathrm{m}\ (\mathrm{m}^2/\mathrm{s})^a\ \mathrm{s}^b\ (\mathrm{mK})^c] = 0.$$

This leads to the following system of equations

$$\begin{aligned}
\text{for m} &: \ 0 = 2a + c + 1,\\
\text{for s} &: \ 0 = -a + b,\\
\text{for K} &: \ 0 = c,
\end{aligned}$$

Diffusion: Dimensional analysis and similarity solution

The solution is $a = -\frac{1}{2}$, $b = -\frac{1}{2}$, and $c = 0$. The first dimensionless group is

$$\Pi_1 = \frac{x}{\sqrt{\alpha t}}.$$

To get $[\Pi_2] = 0$, we must have

$$[\mathrm{K}\ (\mathrm{m}^2/\mathrm{s})^{a'}\ \mathrm{s}^{b'}\ (\mathrm{mK})^{c'}] = 0,$$

which leads to the following system of equations

$$\begin{aligned} \text{for m} &: \ 0 = 2a' + c', \\ \text{for s} &: \ 0 = -a' + b', \\ \text{for K} &: \ 0 = c' + 1. \end{aligned}$$

Diffusion: Dimensional analysis and similarity solution

The solution is $a' = \frac{1}{2}$, $b' = \frac{1}{2}$, and $c' = -1$. The second dimensionless group is

$$\Pi_2 = \frac{T\sqrt{\alpha t}}{V}.$$

Dimensional analysis leads us to pose the solution in the form $\Pi_2 = F(\Pi_1)$. We substitute T into the PDE, with T defined by

$$T = \frac{V}{\sqrt{\alpha t}} F(\xi),$$

with $\xi = x/\sqrt{\alpha t}$. We get

$$\frac{\partial T}{\partial t} = -\frac{1}{2}\frac{V}{2t^{3/2}\sqrt{\alpha}} F(\xi) - \frac{1}{2}\xi \frac{V}{t^{3/2}\sqrt{\alpha}} F'(\xi)$$

$$\frac{\partial T}{\partial x} = \frac{V}{t\alpha} F'(\xi) \text{ and } \frac{\partial^2 T}{\partial x^2} = \frac{V}{(t\alpha)^{3/2}} F''(\xi),$$

Diffusion: Dimensional analysis and similarity solution

This leads us to write the heat equation in the form of a second-order ordinary differential equation

$$-\frac{1}{2}F - \frac{1}{2}\xi F' = F'',$$

which is easy to integrate

$$\frac{1}{2}\xi F + F' = a_0,$$

with a_0 a constant of integration. If propagation occurs in both directions, then the solution is even (heat spreads equally in both directions), and for $\xi = 0$, $F' = 0$ (horizontal tangent), so $a_0 = 0$.

Diffusion: Dimensional analysis and similarity solution

We get

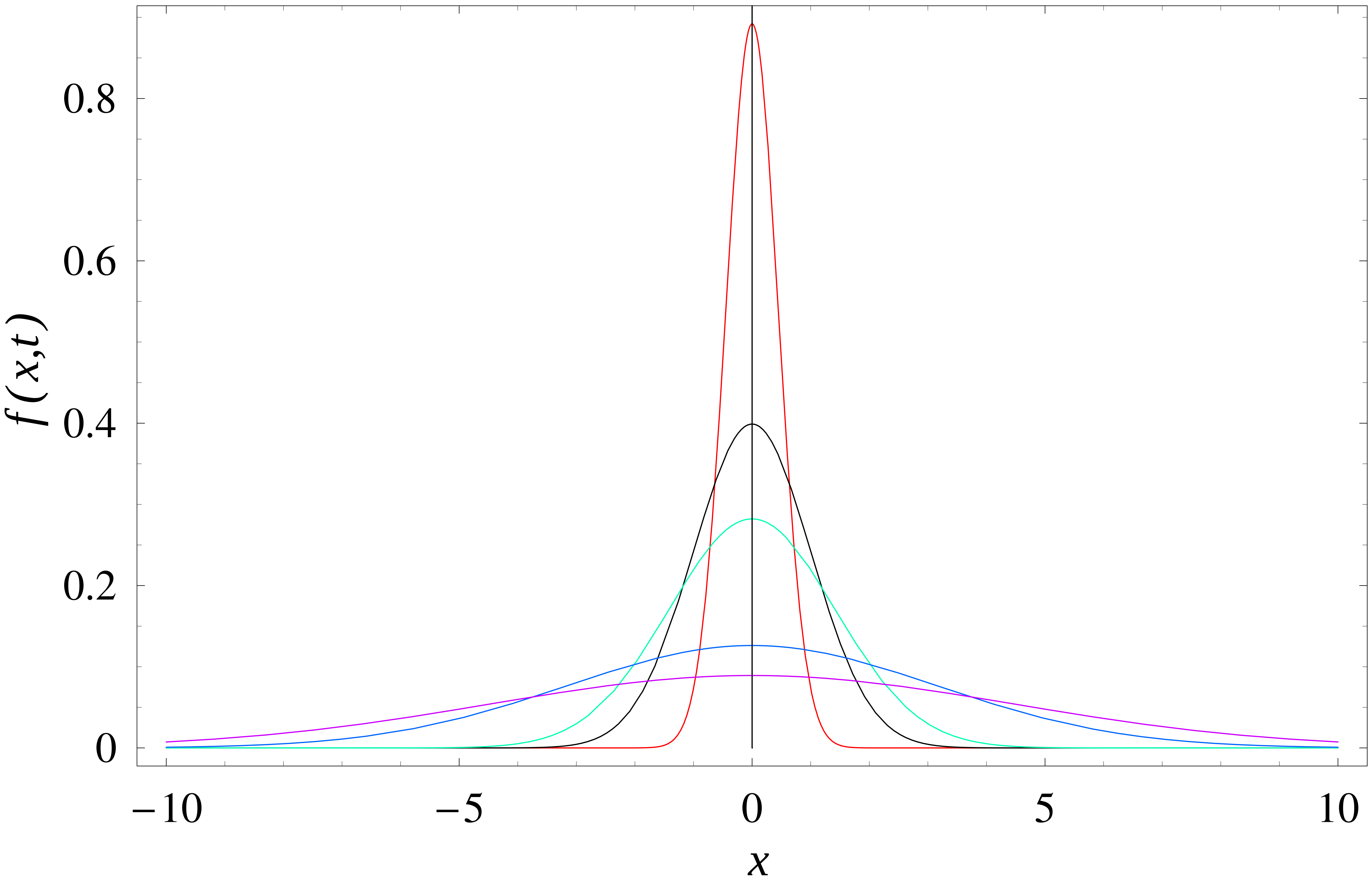
$$\frac{F'}{F} = -\frac{1}{2}\xi \Rightarrow F = a_1 \exp\left(-\frac{1}{4}\xi^2\right)$$

with a_1 a constant of integration. Using the conservation of heat and since $\int F \mathrm{d}\xi = 1$, we deduce $a_1 = 1/(2\sqrt{\pi})$.

The solution reads

$$T = \frac{V}{2\sqrt{\pi \alpha t}} \exp\left(-\frac{1}{4}\frac{x^2}{\alpha t}\right).$$

Diffusion: Dimensional analysis and similarity solution



Exercise 2

2. Consider the diffusion equation:

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2},$$

with D the diffusion coefficient (which is constant).

Exercise 2 (continued)

It is subject to the following initial and boundary conditions

$$f(x,\ 0) = 0,$$
$$f(0,\ t) = a \text{ for } t > 0,$$
$$f(x,\ t) = 0 \text{ for } x \to \infty \text{ and } t > 0,$$

with a a constant.

Solve it using the Laplace transform in t

$$\hat{f}(x,\ s) = \int_0^\infty e^{-st} f(x,\ t)\mathrm{d}t.$$

Advection-diffusion equation

Convection-diffusion is a combination of two phenomena. The linear case is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2},$$

where D and u are assumed constant. This equation can be reduced to a linear diffusion problem by making the following change of variable

$$\zeta = x - ut,$$
$$\tau = t.$$

Advection-diffusion equation

We get

$$
\begin{aligned}
\frac{\partial\cdot}{\partial x} &= \frac{\partial\cdot}{\partial\zeta}\frac{\partial\zeta}{\partial x} + \frac{\partial\cdot}{\partial\tau}\frac{\partial\tau}{\partial x}, \\
&= \frac{\partial\cdot}{\partial\zeta}, \\
\frac{\partial\cdot}{\partial t} &= \frac{\partial\cdot}{\partial\zeta}\frac{\partial\zeta}{\partial t} + \frac{\partial\cdot}{\partial\tau}\frac{\partial\tau}{\partial t}, \\
&= -u\frac{\partial\cdot}{\partial\zeta} + \frac{\partial\cdot}{\partial\tau}.
\end{aligned}
$$

and so

$$
\frac{\partial f}{\partial\tau} = D\frac{\partial^2 f}{\partial\zeta^2},
$$

Advection-diffusion equation: Burgers equation

A special case of convection-diffusion is the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2},$$

which can also be transformed into a diffusion equation using the Cole-Hopf transformation:

$$u = -\frac{2D}{\phi}\frac{\partial \phi}{\partial x},$$

with $\phi(x,\, t)$ an auxiliary function.

Advection-diffusion equation: Burgers equation

We have

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= -\frac{2D}{\phi}\frac{\partial^2\phi}{\partial x^2} + \frac{2D}{\phi^2}\left(\frac{\partial\phi}{\partial x}\right)^2, \\
\frac{\partial u}{\partial t} &= -\frac{2D}{\phi}\frac{\partial^2\phi}{\partial x\partial t} + \frac{2D}{\phi^2}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}, \\
\frac{\partial^2 u}{\partial x^2} &= -\frac{2D}{\phi}\frac{\partial^3\phi}{\partial x^3} - \frac{4D}{\phi^3}\left(\frac{\partial\phi}{\partial x}\right)^3 + \frac{6D}{\phi^2}\frac{\partial^2\phi}{\partial x^2}\frac{\partial\phi}{\partial x}.
\end{aligned}
$$

Advection-diffusion equation: Burgers equation

After simplification, we obtain

$$\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} - \phi\frac{\partial^2\phi}{\partial x\partial t} + 2D\left(\phi\frac{\partial^3\phi}{\partial x^3} - \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}\right) = 0,$$

which can be transformed—by dividing it by ϕ^2, then integrating with respect to x, and ultimately by multiplying it again by ϕ—into a linear diffusion equation

$$\frac{\partial\phi}{\partial t} = D\frac{\partial^2\phi}{\partial x^2}.$$

Wave equation

Dynamic waves are solutions to a differential equation such as:

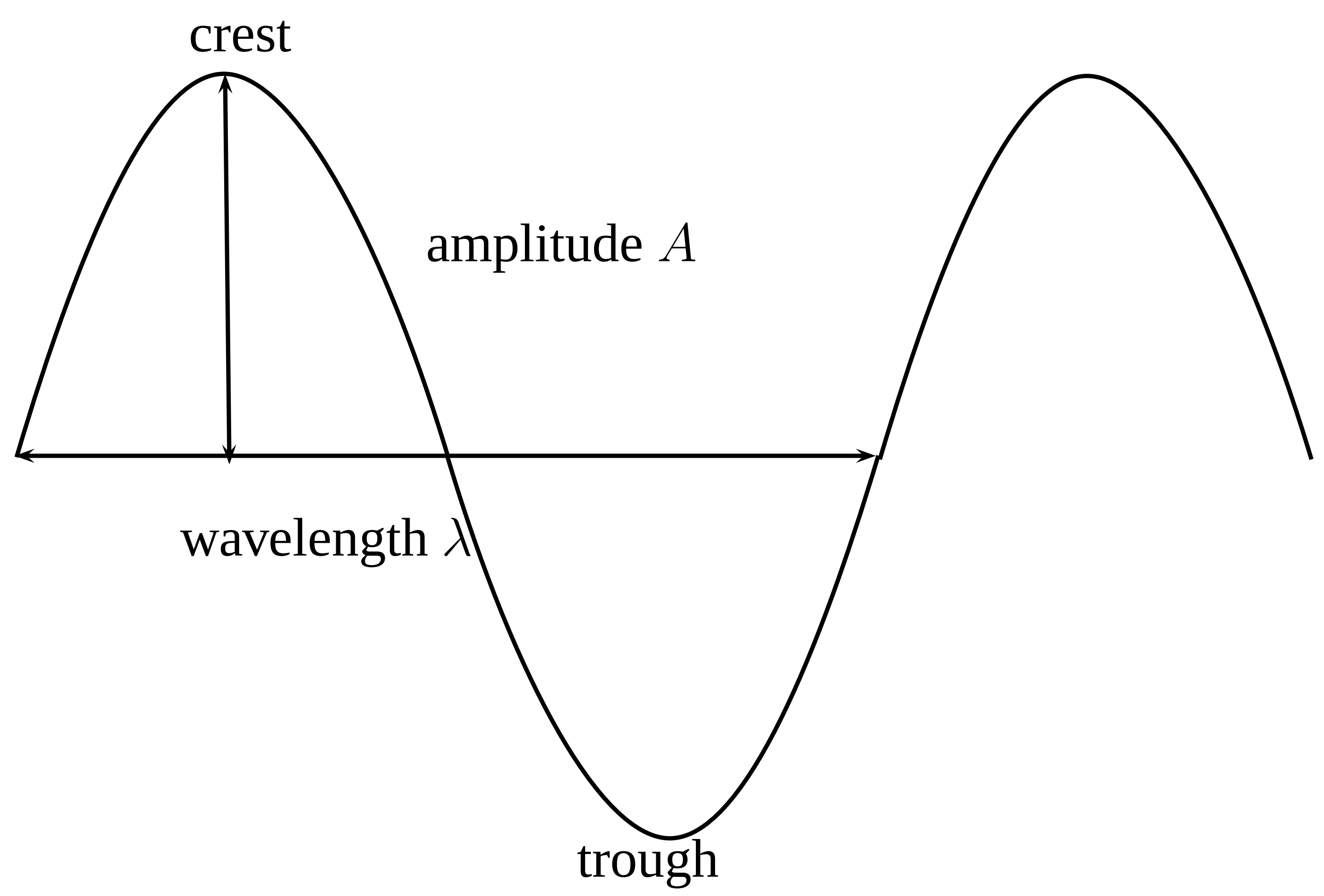
$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2},$$

with c the (phase) velocity. This form is not exhaustive. For example, the equation of gravity waves reads

$$\frac{\partial^2\phi}{\partial t^2} = -g\frac{\partial\phi}{\partial y},$$

with here ϕ the velocity potential $(\mathbf{u}(x,\, y,\, t) = \nabla\phi)$ and g gravity acceleration.

Wave equation



Shape: A is the *amplitude*, k *wave number* ($\lambda = 2\pi/k$ *wavelength*), ω *angular frequency*; we also introduce a frequency f defined as $f = \omega/(2\pi)$: the number of complete oscillations during a second at a given position. The period is defined as $T = \lambda/c$.

Wave equation

Solutions are sought in the form of harmonics (periodic wave)

$$\phi(t) = A\exp[\imath(kx - \omega t)] = \mathrm{Re}(A)\cos(kx - \omega t) - \mathrm{Im}(A)\sin(kx - \omega t),$$

The *wave velocity* is

$$c = \omega/k.$$

The dispersion relation $\omega(k)$ is here linear ($\omega(k) = ck$), i.e. the wave crests move at a constant speed regardless of the wavelength. The phase velocity c_p

$$c_p = \frac{\omega(k)}{k} = c.$$

Wave equation

Note: In a physical process where waves result from the superposition of many harmonic waves of different wavelength, each harmonic component moves at its own speed, which ultimately leads to a separation or *dispersion* of the wave, hence the name *dispersion relation* for $\omega(k)$.

There is a third velocity, called *group velocity*, which represents the speed at which the energy associated with the wave propagates:

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c$$

for linear waves. In general for most physical processes, we have $c_g \leq c_p$.

Wave equation

There are two directions of propagation:

- forward wave $f = f(x - ct)$: the wave goes in the $x > 0$ direction;
- backward wave $f = f(x + ct)$: the wave goes in the $x < 0$ direction.

The general solution to the wave equation reads

$$f = a(x - ct) + b(x + ct),$$

with a and b two functions. This is the *d'Alembert* solution.

Exercise 3

3. Calculate the phase velocity and group velocity of the following equation:

$$u_t + u_x + u_{xxx} = 0.$$

Classification of second-order partial differential equations

The only general classification of partial differential equations concerns linear equations of second order. These equations are of the following form

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g,$$

where a, b, c, d, e, f, and g are real-valued functions x and y. When $g = 0$, the equation is said to be *homogeneous*. Linear equations are classified depending on the sign of $\Delta = b^2 - ac > 0$.

Classification of second-order partial differential equations

If $\Delta = b^2 - ac > 0$, the PDE is *hyperbolic*. The wave equation is an example. In fluid mechanics, transport equations are often hyperbolic. The *canonical* form is

$$u_{xx} - u_{yy} + \cdots = 0 \text{ or, equivalently, } u_{xy} + \cdots = 0,$$

where dots represent terms related to u or its first-order derivatives;

If $\Delta = b^2 - ac < 0$, the PDE is *elliptic*. The Laplace equation is an example. Equations describing equilibrium of a process are often elliptic. The canonical form is

$$u_{xx} + u_{yy} + \cdots = 0$$

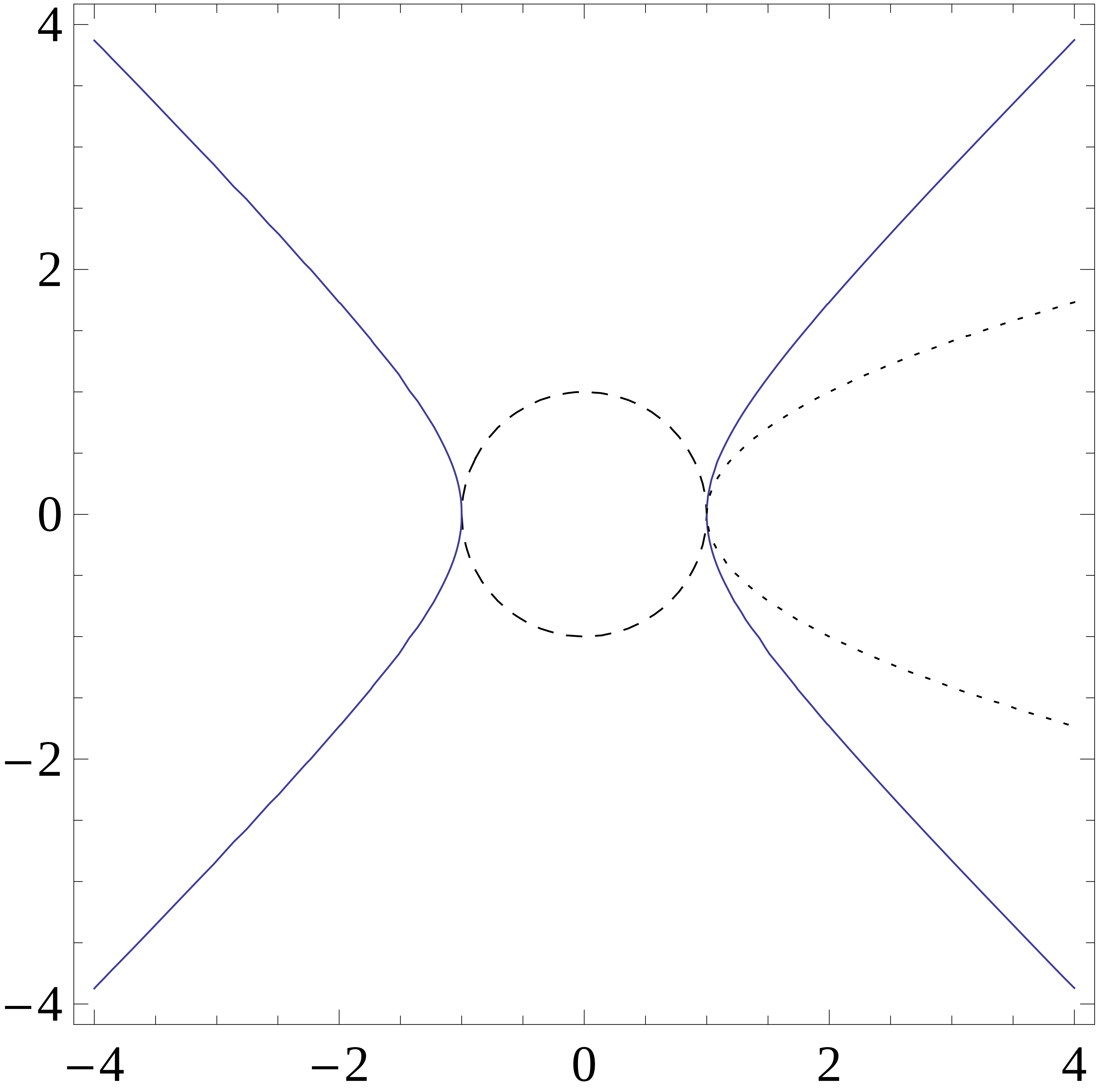
Classification of second-order partial differential equations

If $\Delta = b^2 - ac = 0$, the PDE is *parabolic*. The heat equation is an example. Diffusion equations are often parabolic. The canonical form is

$$u_{yy} + \cdots = 0.$$

Link between the classification and conics: If we substitute u_{xx} with x^2, u_x with x, u_{yy} with y^2, u_y with y, and u_{xy} with xy into the PDE, we obtain the general equation of a conic, which depending on the sign $\Delta = b^2 - ac$ gives a parabola ($\Delta = 0$), an ellipse ($\Delta < 0$), or a hyperbola ($\Delta > 0$).

Classification of second-order partial differential equations



The differential terms are linked and vary according to the constraints intrinsic to each type of curve. For example, for hyperbolic equations, there are two branches and part of the $x - y$ plane is not crossed by the curve, which allows for discontinuous jumps from one branch to another: a hyperbolic equation is able to generate solutions that become discontinuous, i.e. undergo a shock even if initially they were continuous.

Boundaries

In mechanics, we solve equations with space and time variables. In general, in order to work out a particular solution u to a partial differential equation, we need

- boundary conditions that specify how u varies along the domain boundaries at any time;
- the initial conditions that specify how varies u at the initial instant for any point in the domain.

We must solve what is called a boundary-value problem with initial conditions or, said differently, *initial boundary-value problem*. In some cases, we do not need as much information. For example, for certain hyperbolic equations, one needs only the initial conditions, whereas elliptic problems require only boundary conditions (they generally reflect stationary processes).

Boundaries

We distinguish:

- *Dirichlet boundary conditions*: the boundary condition specifies the value u_0 that the function takes at a point or a curve

$$u(\mathbf{x}\ ;\ t) = u_0(t)$$

along a curve Γ.

- *Neuman boundary conditions*: the boundary conditions specify the derivative that the function takes at a point or a series of points. Physically, this reflects a flux condition across the domain boundary:

$$\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{n}\ ;\ t) = \boldsymbol{\phi}(t)$$

along a curve Γ, with $\boldsymbol{n}$ the normal to Γ and $\boldsymbol{\phi}(t)$ a flux function.

Exercise 4

4. Consider the following initial-value problem

$$t\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial t} = 0,$$

with $u(x,\ 0) = f(x)$ for $x > 0$. What type is this equation? Solve it after determining the associate characteristic equation.

Exercises 5 and 6

5. The Euler-Darboux equation reads

$$u_{xy} + \frac{au_x - bu_y}{x - y} = 0.$$

Characterise this equation.

6. The Helmholtz equation reads

$$\nabla^2 u + k^2 u = 0.$$

Characterise this equation.

Exercises

7. The Klein-Gordon equation is a variant of the Schrödinger equation, which described how an elementary particle behaves. It reads

$$\frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} + c^2 u = 0.$$

Characterise this equation. Seek periodic solutions in the form $u(x,\ t) = a(k)\exp(\imath kx + \lambda(k)t)$ with a the amplitude of the wave and where λ and k are the modes. Determine the mode λ? Is the solution stable?

8. Find and sketch the regions in the (x, y)-plane where the equation

$$(1 + x)\phi_{xx} + 2xy\phi_{xy} + y^2\phi_{yy} = 0$$

is elliptic, parabolic, and hyperbolic.