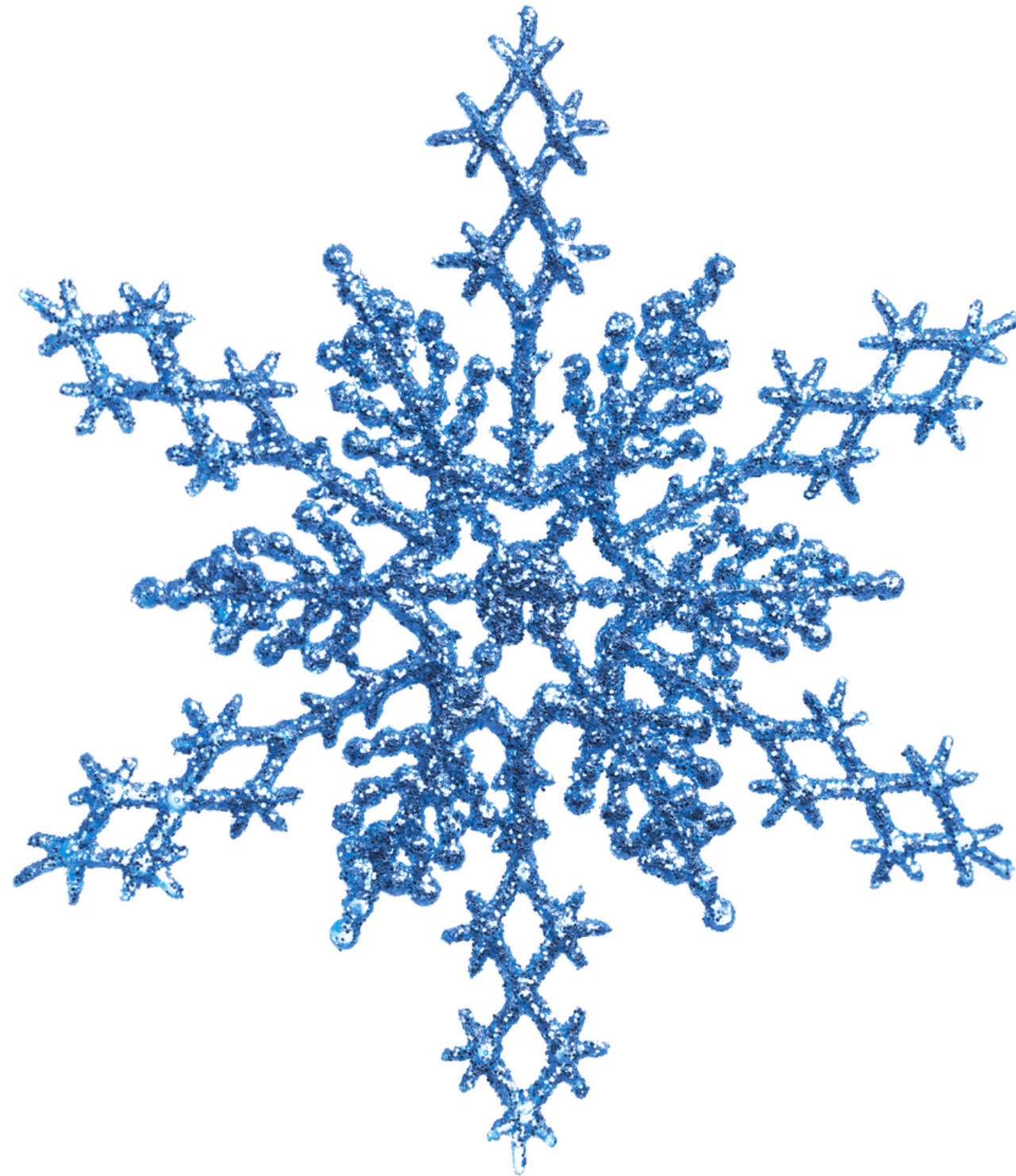


Chapter 3: One-parameter groups (Lie groups)

Similarity and Transport Phenomena in Fluid Dynamics

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Chapter 3: One-parameter groups



- Groups of transformation
- Infinitesimal transformation
- Group invariants
- Invariant curves
- Transformation of derivative

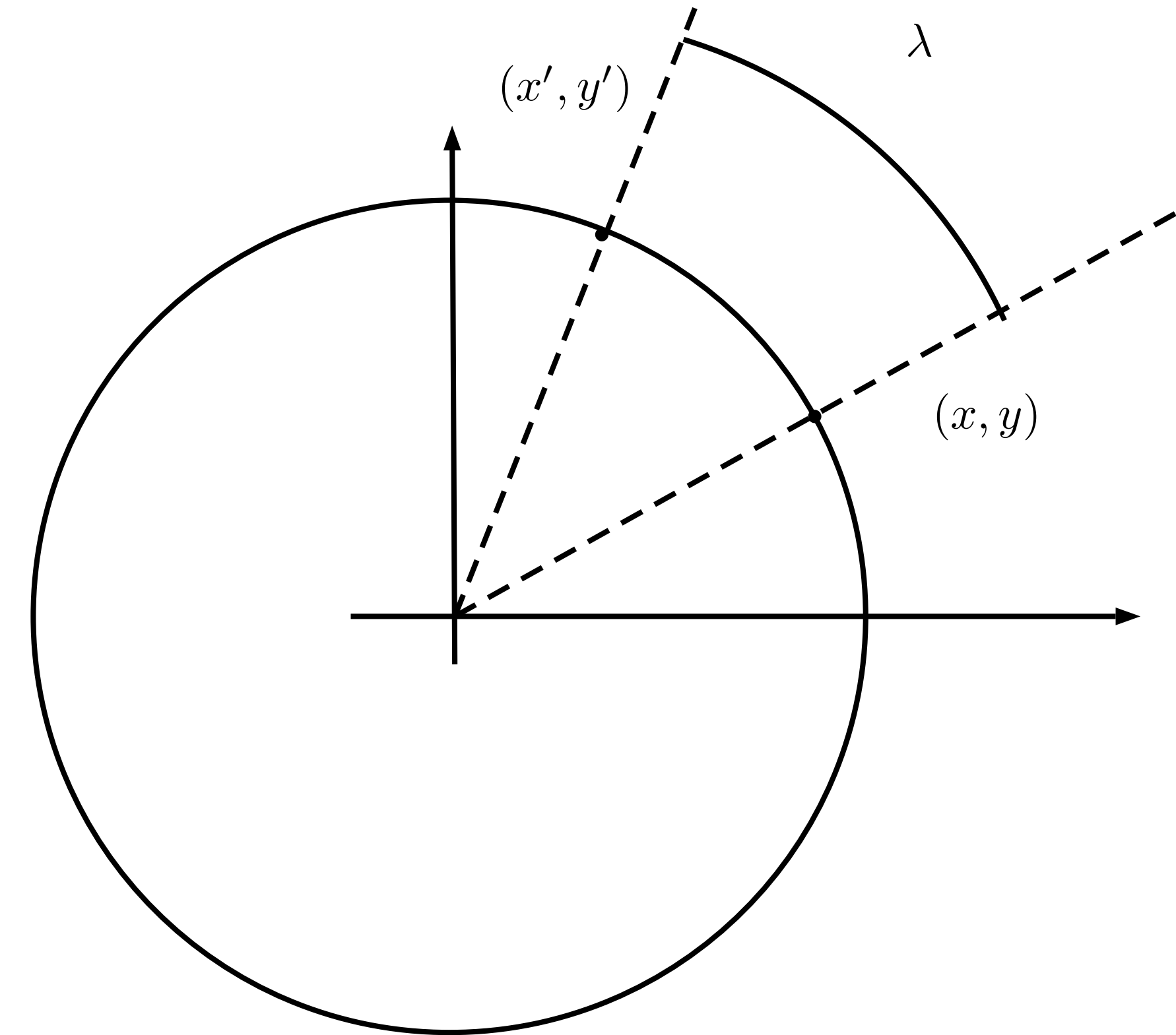
Consider the transformation

$$\begin{aligned}x' &= X(x, y; \lambda), \\y' &= Y(x, y; \lambda),\end{aligned}$$

that depends on the continuous parameter λ . For instance, the rotation around the origin

$$\begin{aligned}x' &= x \cos \lambda - y \sin \lambda, \\y' &= x \sin \lambda + y \cos \lambda,\end{aligned}$$

where λ is the angle of rotation.



Properties of rotations: group properties

- two rotations λ and λ' carried out in succession are equivalent to the rotation $\lambda'' = \lambda + \lambda'$
- If $\lambda' = -\lambda$, then the second image coincides with the source point (*identity transformation*)
- each rotation has an inverse

λ_0 the value of λ for which the transformation is the identity transformation. First order Taylor's series around λ_0

$$x' = X(x, y; \lambda) = x + \left. \frac{\partial X}{\partial \lambda} \right|_{\lambda_0} (\lambda - \lambda_0) + \dots,$$

$$y' = Y(x, y; \lambda) = y + \left. \frac{\partial Y}{\partial \lambda} \right|_{\lambda_0} (\lambda - \lambda_0) + \dots,$$

We introduce the coefficients of the infinitesimal transformation:

$$\xi = \left. \frac{\partial X}{\partial \lambda} \right|_{\lambda_0} \quad \text{and} \quad \eta = \left. \frac{\partial Y}{\partial \lambda} \right|_{\lambda_0}.$$

Working with the group is equivalent to working with its infinitesimal representation (Lie theorem).

First-order expansion = infinitesimal transformation

$$x' = x + \xi(x, y)(\lambda - \lambda_0),$$

$$y' = y + \eta(x, y)(\lambda - \lambda_0)$$

This is the Euler approximation of the coupled differential equations

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)} = d\lambda.$$

The path originating from the source point (x, y) is the locus of all the images of the source point. It is called the *orbit* of the group. The differential operator

$$\Gamma = \xi \partial_x + \eta \partial_y$$

is called the *infinitesimal generator* or the *group operator*.

Exercises 1 and 2



1. Determine the orbit of the rotation group.
2. Does the translation $x' = x + s$ form a group? If so, determine the infinitesimal operator.

A *group invariant* is a function $u(x, y)$ whose value at an image point is the same as its value at the source point

$$u(x', y') = u(x, y) \text{ or } u(X(x, y; \lambda), Y(x, y; \lambda)) = u(x, y).$$

Thus u is constant along an orbit. Differentiating with respect to λ , then setting $\lambda = \lambda_0$, we find

$$\xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} = 0 \Leftrightarrow \frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)} \Leftrightarrow \Gamma u = 0.$$

Any function of the integral of this characteristic equation is an invariant.

An *invariant curve* is one whose points map into other points for all transformations of the group. The curve C is either an orbit of the group or a locus on which the infinitesimal coefficients ξ and η vanish simultaneously. An implicit representation of curves is

$$\phi(x, y) = c,$$

where ϕ is a one-parameter family of curves and c is a parameter. The family is invariant if the image of each curve is another curve of the family, namely

$$\phi(x', y') = c'$$

Let us differentiate $\phi(x', y') = c'$ with respect to λ and set $\lambda = \lambda_0$

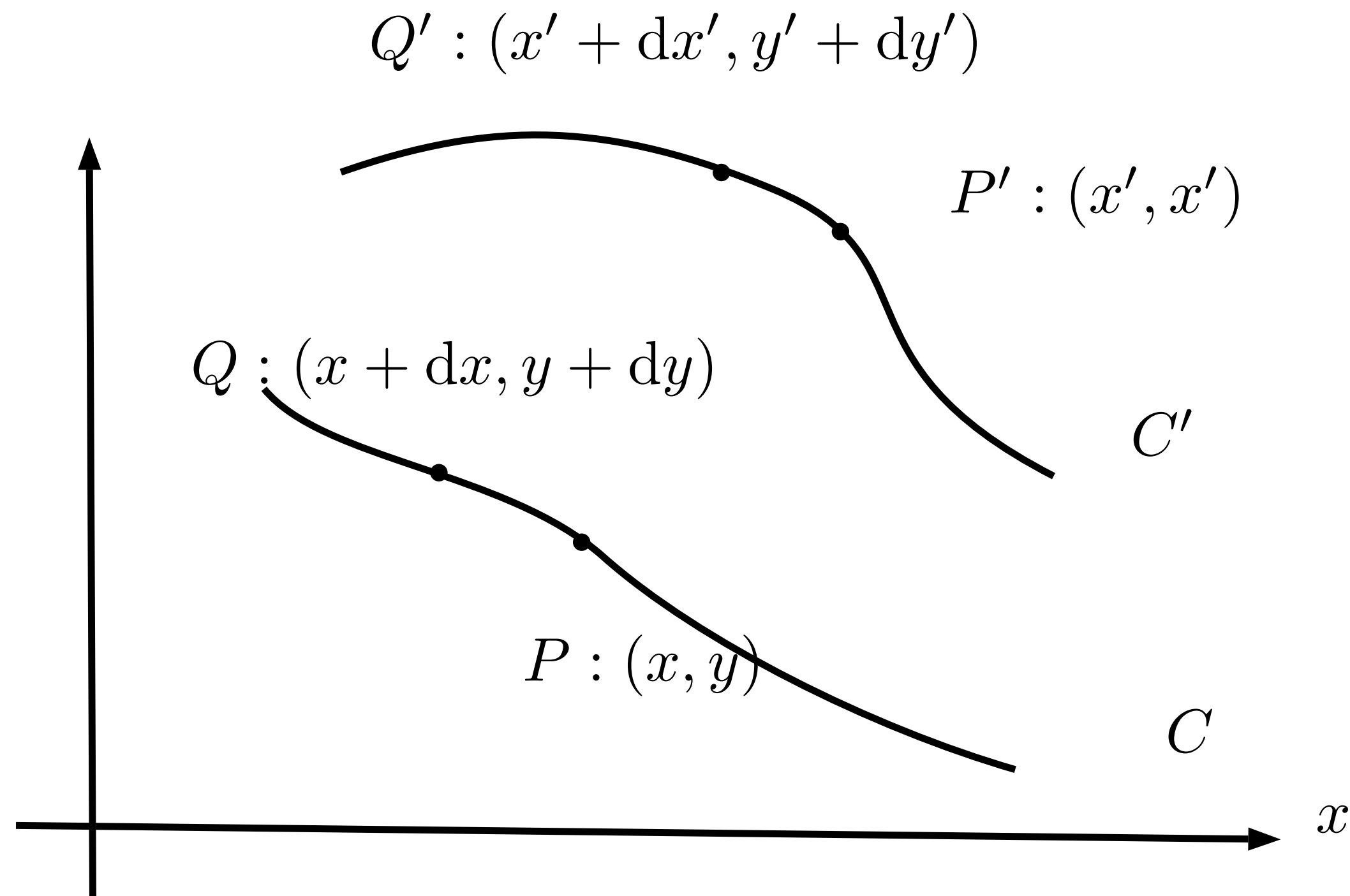
$$\xi \frac{\partial \phi}{\partial x} + \eta \frac{\partial \phi}{\partial y} = \left. \frac{\partial c'}{\partial \lambda} \right|_{\lambda=\lambda_0} .$$

The right-hand side is a function of c alone (let us call it F). Its choice is up to us (we then obtain different families). For instance, we can take $F = 1$.

Summary:

- $\Gamma u = 0$ is the equation for invariant curves.
- $\Gamma u = 1$ is the equation of invariant families.

Extended group



Question: $C \rightarrow C'$, how to calculate $y' = dy'/dx'$ from $y = dy/dx$?

Coordinates of P'

$$x' = X(x, y; \lambda)$$

$$y' = Y(x, y; \lambda)$$

and those of Q'

$$x' + dx' = X(x + dx, y + dy; \lambda)$$

$$y' + dy' = Y(x + dx, y + dy; \lambda)$$

To leading order, we get

$$dx' = X_x dx + X_y dy$$

$$dy' = Y_x dx + Y_y dy$$

whose ratio is

$$\dot{y}' = \frac{dy'}{dx'} = \frac{Y_x dx + Y_y dy}{X_x dx + X_y dy}.$$

The transformation $(x, y, \dot{y}) \rightarrow (x', y', \dot{y}')$ forms a group called the *once-extended group*. What is the coefficient of the infinitesimal transformation?

Using the infinitesimal form, we have

$$dx' = dx + d\xi(\lambda - \lambda_0)$$

$$dy' = dy + d\eta(\lambda - \lambda_0)$$

We then find that the slope of the image curve is

$$y' = \frac{dy'}{dx'} = \frac{\frac{dy}{dx} + \frac{d\eta}{dx}(\lambda - \lambda_0)}{1 + \frac{d\xi}{dx}(\lambda - \lambda_0)}.$$

To leading order, we find

$$y' = \frac{dy'}{dx'} = \frac{dy}{dx} + \left(\frac{d\eta}{dx} - \frac{dy}{dx} \frac{d\xi}{dx} \right) (\lambda - \lambda_0).$$

The coefficient is then

$$\eta_1 = \frac{d\eta}{dx} - \dot{y} \frac{d\xi}{dx}$$

with the total derivatives (directional derivatives in the direction \dot{y})

$$\frac{d\eta}{dx} = \eta_x + \eta_y \dot{y}$$

$$\frac{d\xi}{dx} = \xi_x + \xi_y \dot{y}$$

Higher-order derivatives are obtained by iteration ($k \geq 1$)

$$\eta_{k+1} = \frac{d\eta_k}{dx} - \dot{y}^{(k+1)} \frac{d\xi}{dx}$$

Invariant differential equations of the first order

An invariant of the once-extended group is a function $u(x, y, \dot{y})$ that satisfies

$$u(x, y, \dot{y}) = u(x', y', \dot{y}')$$

Upon differentiation, we show that u satisfies

$$\xi u_x + \eta u_y + \eta_1 u_{\dot{y}} = 0,$$

or

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)} = \frac{d\dot{y}}{\eta_1(x, y, \dot{y})}.$$

This extension has interesting applications that we will see in the next chapters.



3. Consider the stretching group

$$x' = \lambda x,$$

$$y' = \lambda^\beta y$$

Determine the coefficients of the infinitesimal transformation and the orbit.