ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 5: Second-order differential equations

Similarity and Transport Phenomena in Fluid Dynamics

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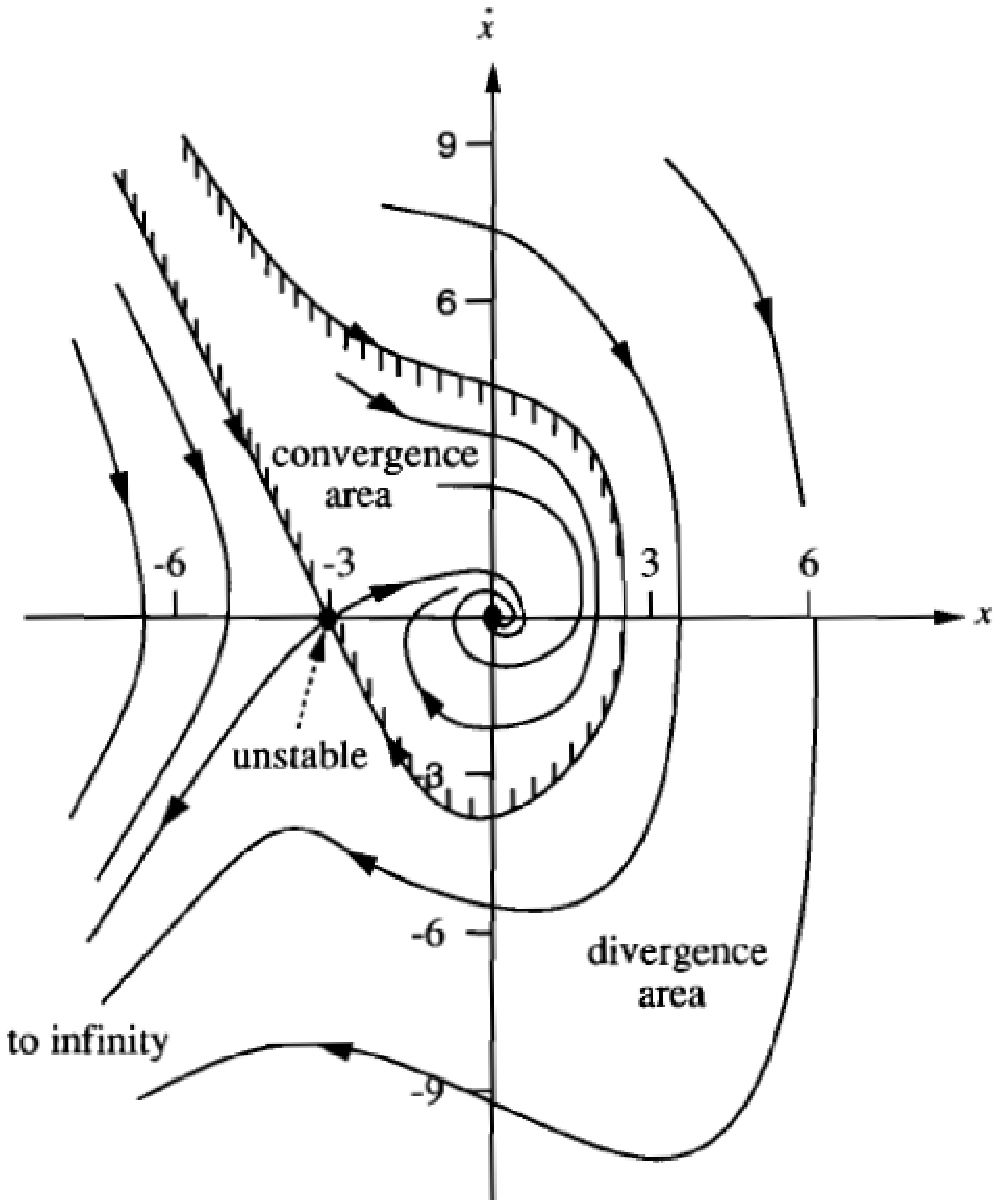
Chapter 5: Second-order differential equations



- Invariant differential equations
- Lie's reduction theorem
- Stretching group
- Determining equations

Twice-extended group

An invariant of the twice-extended group is a function $w(x, y, \dot{y}, \ddot{y})$ whose value at an image point is the same as its value at the source point

$$w(x', y', \dot{y}', \ddot{y}') = w(x, y, \dot{y}, \ddot{y})$$

We follow the same procedure as for Chapter 4: differentiate with respect to λ, then set $\lambda = \lambda_0$. We get

$$\xi w_x + \eta w_y + \eta_1 w_{\dot{y}} + \eta_2 w_{\ddot{y}} = 0$$

whose characteristic equations are

$$\frac{\mathrm{d}x}{\xi(x, y)} = \frac{\mathrm{d}y}{\eta(x, y)} = \frac{\mathrm{d}\dot{y}}{\eta_1(x, y, \dot{y})} = \frac{\mathrm{d}\ddot{y}}{\eta_2(x, y, \dot{y}, \ddot{y})}$$

Lie's reduction theorem

A second-order differential equation can be written in the form

$$w(x, y, \dot{y}, \ddot{y}) = 0$$

It can also be written as a pair of coupled differential equations of the first-order by setting $u = \dot{y}$. The pair of equations must then be solved *simultaneously*

$$u = \dot{y}$$

$$w(x, y, u, \dot{u}) = 0$$

Lie has shown how to reduce the problem of solving this system to that of solving two first-order equations one at a time *successively*.

Lie's reduction theorem

If we start from

$$u = \dot{y}$$
$$w(x, y, u, \dot{u}) = 0$$

we can plot curves in the space (x, y, u). If we set $\mathrm{d}x$ at a point (x, y, u), then we solve the system of equations to deduce $\mathrm{d}y = u\mathrm{d}x$ and $\mathrm{d}u = \dot{u}\mathrm{d}x$. The curves thus depend on two parameters (e.g., the constants of integration) and form a two-parameter family.

Let us assume that we know a group whose infinitesimal coefficients are ξ and η and that leaves the system invariant (once it has been extended). The transformations of the group carry each of the curves of the two-parameter family to another curve of the family.

Lie's reduction theorem

The invariant surfaces form a one-parameter family

$$\phi(x, y, u, c) = 0$$

with c a parameter. The invariance of each surface of this family implies that

$$\phi(x', y', u', c') = \phi(X(x, y; \lambda), Y(x, y; \lambda), U(x, y, u; \lambda), c) = 0$$

Upon differentiation with respect to λ, we get

$$\xi\phi_x + \eta\phi_y + \eta_y\phi_u = 0 \Leftrightarrow \frac{\mathrm{d}x}{\xi(x, y)} = \frac{\mathrm{d}y}{\eta(x, y)} = \frac{\mathrm{d}u}{\eta_1(x, y, u)}$$

If $p(x, y)$ and $q(x, y, u)$ are two integrals of the characteristic equations, then the solution is an arbitrary function G of p and q

$$G(p, q, c) = 0$$

Lie's reduction theorem

The function p is a group invariant while q is an invariant of the once-extended group, called the *first-differential invariant*. Lie's reduction theorem states that if we adopt p and q as new variables, then the second-order differential equation $w(x, y, \dot{y}, \ddot{y}) = 0$ reduces to a first-order differential equation in p and q. The latter is called the *associated differential equation*.

Example: Emden-Fowler equation

The Emden-Fowler equation

$$\ddot{y} + 2\frac{\dot{y}}{x} + y^n = 0$$

arises in astrophysics. We assume that $n \neq 1$. It is a particular case of the Thomas–Fermi equation in quantum mechanics. This equation is invariant to the twice-extended stretching group

$$x' = \lambda x$$

$$y' = \lambda^{\beta} y$$

$$\dot{y}' = \lambda^{\beta-1} \dot{y}$$

$$\ddot{y}' = \lambda^{\beta-2} \ddot{y}$$

Example: Emden-Fowler equation

Substitution into the Emden-Fowler equation gives $\beta = 2/(1-n)$. An invariant p and a first differential invariant q are thus

$$p = \frac{y}{x^\beta} \text{ and } q = \frac{\dot{y}}{x^{\beta-1}}$$

Note that this choice is not unique: any combination of p and q is also a first differential invariant. Let us now calculate the derivatives

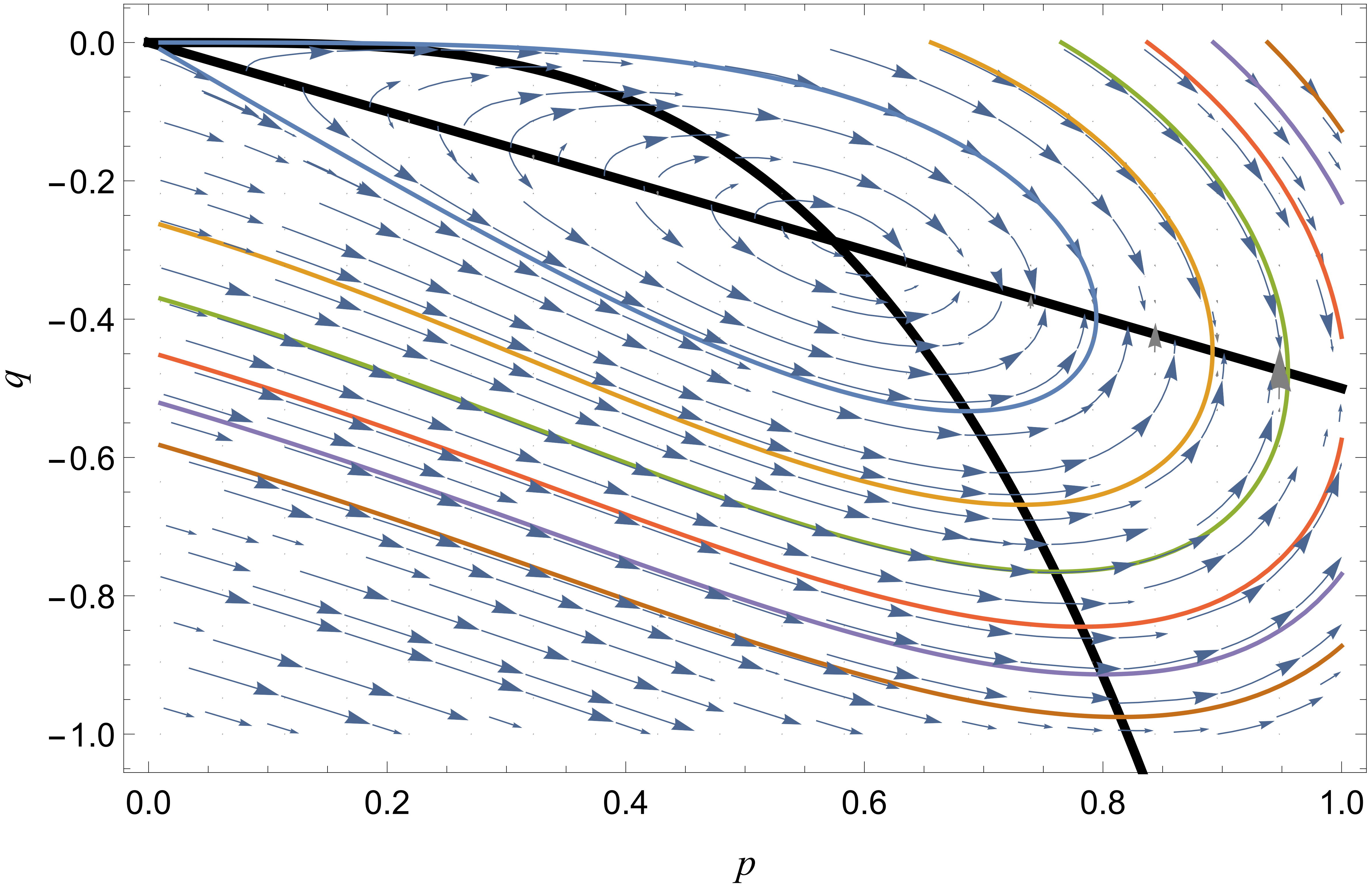
$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\dot{y}}{x^\beta} - \beta\frac{y}{x^{\beta+1}} = \frac{q-\beta p}{x} \text{ and } \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\ddot{y}}{x^{\beta-1}} - (\beta-1)\frac{\dot{y}}{x^\beta} = -\frac{(\beta+1)q + p^{\beta-2/\beta}}{x}$$

The derivative ratio is

$$\frac{\mathrm{d}q}{\mathrm{d}p} = -\frac{(\beta+1)q + p^{\beta-2/\beta}}{q-\beta p}$$

So we have transformed a second-order ODE into a first-order ODE.

Example: Emden-Fowler equation



Example of phase portrait for $\beta = -1/2$

Example: Emden-Fowler equation

The associated differential equation can be cast in the form

$$\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{f(p,q)}{q - \beta p}$$

Two curves (*nullclines*) emerge:

- the locus $f(p,q) = 0$ where $q(p)$ admits a zero derivative (tangent horizontal)
- the locus $q - \beta p = 0$ where $q(p)$ admits an infinity derivative (tangent vertical)

Two singular points P that are intersections of these curves:

- origin P = O: trivial solution $p = q = 0$
- point P (let us call it Q): asymptotic solution $y = Ax^\beta$ (check this).

Example: Emden-Fowler equation

The behaviour of the singular point P $(r_P \;\; v_P)$ is evaluated by linearizing the equations near P:

$$x\frac{\mathrm{d}q}{\mathrm{d}x} = f(p,q) = 0 + (p - p_P)\partial_p f + (q - q_P)\partial_q f + o(p,q)$$
$$x\frac{\mathrm{d}p}{\mathrm{d}x} = (q - q_P) - \beta(p - p_P).$$

We seek solutions in the form $p = p_P + Pe^{\lambda x}$ et $q = q_P + Qe^{\lambda x}$, then λ is an eigenvalue of:

$$\begin{bmatrix} -\beta & 1 \\ \partial_p f & \partial_q f \end{bmatrix}$$

and so we get

$$\lambda = \frac{\partial_q f - \beta}{2} \pm \frac{\sqrt{(\beta + \partial_q f)^2 + 4\partial_p f}}{2}.$$

Example: Emden-Fowler equation

Reminder

- If the two eigenvalues are real and of same sign, then the point is a node
- If the two eigenvalues are real and of opposite sign, then the point is a saddle point
- If the two eigenvalues are imaginary, then the point is a focal point

Here for $\beta = -1/2$, we have $p_P = (-\beta(1+\beta))^{\beta/(\beta^2-\beta-2))} = 1/4^{2/5}$.

$$\begin{bmatrix} -\beta & 1 \\ \partial_p f & \partial_q f \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ -7/8 & -1/2 \end{bmatrix}$$

The two eigenvalues are imaginary, so Q is a focal point (solutions wrap around Q, so no asymptotic behaviour!). The same exercise for O shows it is associated with two real eigenvalues of opposite sign: O is a saddle point.

Example: Emden-Fowler equation

What is the asymptotic behaviour of the solution near P=O? The eigenvectors are $\mathbf{e}_1 = (1,0)$ et $\mathbf{e}_2 = (1,-1)$, i.e. the slope of the solution is $m=0$ or $m=-1$ (Making use of the L'Hospital rule leads to the same result. Check it). There are two possibilities here: $m=0$ or $m=-1$. To leading order, we have

$$q - q_P = m(p - P_P)$$

and since $x\mathrm{d}p/\mathrm{d}x = q - \beta p$, we deduce

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}p}{(m-\beta)(p-p_P)},$$

and upon integration

$$x \approx (p - p_p)^{1/(m-\beta)},$$

or equivalently $y(x) \approx \left(p_p + x^{m-\beta}\right) x^{\beta}$.

Example: Emden-Fowler equation

We infer that if $m > \beta$ then x tends to infinity when $p \to p_P$. On the opposite, if $m < \beta$ then x tends to 0 when $p \to p_P$. We know that y takes finite values. We must have $m = 0$ as $x \to 0$. When $x \to \infty$ then $m \to -1$: $y(x) \propto x^m \approx x^{-1}$. In the (p, q) plane, the solution is one of the curves plotted clockwise (starting from O and returning to O).

Remark. When $\beta = -1/2$ ($n = 5$), the associated differential equation gives

$$\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{f(p,q)}{q - \beta p} \Rightarrow q^2 + qp + \frac{4}{9}p^{9/2} = a,$$

with a a constant of integration. The original second-order ODE is equivalent to

$$\ddot{y} + 2\frac{\dot{y}}{x} + y^5 = 0 \Leftrightarrow \left(\frac{\dot{y}}{x^{-3/2}}\right)^2 + \frac{\dot{y}}{x^{-3/2}}\frac{y}{x^{-1/2}} + \frac{4}{9}\frac{y^{9/2}}{x^{-9/4}} = a$$

Example: Emden-Fowler equation

Reminder: (see Chap. 4, Theorems 2 and 3). If we have $X(x, y; \lambda) = \lambda x$ (the transformation of x is a stretching) then $\xi = x$. We can satisfy theorem 2 by setting $F = x$. For groups in which the transformation of x is a stretching, introducing a group invariant as new variable in place of y and keeping x leads to a new ODE that is separable.

An invariant of the stretching group satisfies

$$\xi G_x + \eta G_y = 0 \text{ with } \xi = x \text{ with } \eta = \beta y$$

Here the characteristic equations are

$$\frac{\mathrm{d}x}{x} = -2\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}G}{0}$$

Any function of $s = y^2 x$ is a group invariant.

Example: Emden-Fowler equation

So we change the change of variable $(x, y) \to (x, s = y^2 x)$.

$$\frac{4}{9} y^4 \sqrt{y\sqrt{x}} + y\dot{y} + x\dot{y}^2 \Rightarrow -s + \frac{16}{9} s^{-9/4} + x^2 \frac{\dot{s}^2}{s} = 0$$

that can be in the separable form

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}s}{\sqrt{s - \frac{16}{9} s^{9/4}}}$$

which can be integrated (e.g., with Mathematica).

Stretching groups: asymptotic behaviour

Let us assume that the ODE $w(x, y, \dot{y}, \ddot{y})$ is invariant to the stretching group

$$x' = \lambda x \text{ and } y' = \lambda^{\beta} y,$$

with β a constant. The infinitesimal coefficients are

$$\xi = x, \eta = \beta y, \eta_1 = (\beta - 1)\dot{y} \text{ and } \eta_2 = (\beta - 2)\ddot{y}$$

The most general differential invariant to this group is a function ϕ of the three integrals of the characteristic equations

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{\beta y} = \frac{\mathrm{d}\dot{y}}{(\beta - 1)\dot{y}} = \frac{\mathrm{d}\ddot{y}}{(\beta - 2)\ddot{y}}$$

Stretching groups: asymptotic behaviour

The functions y/x^β, $\dot{y}/x^{\beta-1}$, and $\ddot{y}/x^{\beta-2}$ are three such integrals, and thus

$$\phi(y/x^\beta, \dot{y}/x^{\beta-1}, \ddot{y}/x^{\beta-2}) = 0$$

A power-law function $y = Ax^\beta$ satisfies this equation. We therefore deduce that A is solution to the algebraic equation

$$\phi(A, (\beta-1)A, (\beta-2)A) = 0$$

One of the solutions gives the asymptotic behaviour of y.

Exercise 1: Thomas-Fermi equation

1. Consider the differential equation
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{y^{3/2}}{\sqrt{x}},$$
subject to $y(0) = 1$ and $y(\infty) = 1$. (a) Show that this ODE is invariant to a stretching group. (b) By making use of the change of variables $(x, y) \to (p, q)$, determine the associated differential equation and plot the phase portrait. (c) Determine the asymptotic solution to this ODE. (d) How it is represented in the phase portrait. (e) How can you solve the associated differential equation numerically?

Determining equations

Let us consider the ODE

$$\ddot{y} = \omega(x, y, \dot{y},)$$

and the twice-extended infinitesimal operator

$$\Gamma^{(2)} = \xi\partial_x + \eta\partial_y + \eta_1\partial_{\dot{y}} + \eta_2\partial_{\ddot{y}}$$

with

$$\eta_1 = \eta_x + \eta_y\dot{y} - \dot{y}(\xi_x + \xi_y\dot{y}) = \eta_x + (\eta_y - \xi_x)\dot{y} - \xi_y\dot{y}^2,$$

$$\eta_2 = \mathrm{d}\eta_1/\mathrm{d}x - \ddot{y}\mathrm{d}\xi/\mathrm{d}x = \eta_{xx} + (2\eta_{xy} - \xi_{xx})\dot{y} + (\eta_{yy} - 2\xi_{xy})\dot{y}^2 - \xi_{yy}\dot{y}^3 + (\eta_y - 2\xi_x - 3\xi_y\dot{y})\ddot{y}.$$

This operator leaves the ODE invariant provided that

$$\Gamma^{(2)}(\ddot{y} - \omega) = 0$$

Determining equations

The twice-extended infinitesimal operator can be expressed as

$$\Gamma^{(2)} = \xi\partial_x + \eta\partial_y + \left(\eta_x + (\eta_y - \xi_x)\dot{y} - \xi_y\dot{y}^2\right)\partial_{\dot{y}} + \\ \left(\eta_{xx} + (2\eta_{xy} - \xi_{xx})\dot{y} + (\eta_{yy} - 2\xi_{xy})\dot{y}^2 - \xi_{yy}\dot{y}^3 + (\eta_y - 2\xi_x - 3\xi_y\dot{y})\ddot{y}\right)\partial_{\ddot{y}}.$$

The condition $\Gamma^{(2)}(\ddot{y} - \omega) = 0$ yields the equation

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})\dot{y} + (\eta_{yy} - 2\xi_{xy})\dot{y}^2 - \xi_{yy}\dot{y}^3 + (\eta_y - 2\xi_x - 3\xi_y\dot{y})\omega = \\ \xi\omega_x + \eta\omega_y + (\eta_x + (\eta_y - \xi_x)\dot{y} - \xi_y\dot{y}^2)\omega_{\dot{y}}.$$

↬ x, y, $\dot{y}$, et $\ddot{y}$ are considered independent variables. Albeit looking complex, the equation is simple than the original ODE: it is linear and it involves independent infinitesimal coefficients providing the *determining equations*.

Determining equations

For instance, let us consider the ODE

$$\ddot{y} = 0$$

Thus we have $w(w, y,\ \dot{y}) = 0)$. We deduce

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})\dot{y} + (\eta_{yy} - 2\xi_{xy})\dot{y}^2 - \xi_{yy}\dot{y}^3 = 0.$$

As ξ and η are independent de $\dot{y}$, this equation transforms into a system of equations :

$$\eta_{xx} = 0, 2\eta_{xy} = \xi_{xx}, \eta_{yy} = 2\xi_{xy},\ \text{and}\ \xi_{yy} = 0.$$

Determining equations

$\eta = A(x)y + B(x)$, $\xi = A(x)\dot{y}^2 + C(x)y + D(x)$, where A, B, C, D are arbitrary functions. Finally we obtain:

$$\xi = c_1 + c_3 x + c_5 y + c_7 x^2 + c_8 xy,$$
$$\eta = c_2 + c_4 y + c_6 x + c_7 xy + c_8 y^2,$$

with c_i constants.

Tiresome task! There are packages that can find the determining equations and do the work, e.g. with Mathematica

http://web.stanford.edu/∼cantwell/SymmetryAnalysisSoftware/

Homework: superfluid helium

Gorter-Mellink's law predict that for low-temperature helium, heat flux is related to temperature gradient by $q^3 = -k^3 \nabla T$. There are similarity solutions to the resulting nonlinear diffusion. The principal differential equation is given by

$$4\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{5/3} = 0$$

Solve the equation subject to $y(0) = cst > 0$ and $y(\infty) = 0$. Hint: find the groups that leave the ODE invariant. (Note: the notation is awkward as we calculate $\dot{y}^{5/3}$ with $\dot{y} < 0$; here assume that $(-1)^{5/3} = -1$).