ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 8: Hyperbolic partial differential equations

Similarity and Transport Phenomena in Fluid Dynamics

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Chapter 8: Hyperbolic partial differential equations

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Hyperbolic problems

Hyperbolic problems arise frequently in fluid mechanics (and continuum mechanics). For instance, in hydraulic engineering:

Dimension 1: nonlinear convection equation, for example the kinematic wave equation, which describes flood propagation in rivers

$$\frac{\partial h}{\partial t} + K\sqrt{i}\frac{\partial h^{5/3}}{\partial x} = 0,$$

with h flow depth, K Manning-Strickler coefficient, et i bed gradient;

Dimension 2: Saint-Venant equations (also called the shallow water equations)

$$\frac{\partial h}{\partial t} + \frac{\partial h\bar{u}}{\partial x} = 0,$$
$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} = g\sin\theta - g\cos\theta\frac{\partial h}{\partial x} - \frac{\tau_p}{\varrho h},$$

with $\bar{u}$ flow-depth averaged velocity, h flow depth, θ bed slope, τ_p bottom shear stress.

Hyperbolic problems

Dimension 3: Saint-Venant equations with advection of pollutant

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial h\bar{u}}{\partial x} &= 0,\\
\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} &= g\sin\theta - g\cos\theta\frac{\partial h}{\partial x} - \frac{\tau_p}{\varrho h},\\
\frac{\partial \varphi}{\partial t} + \bar{u}\frac{\partial \varphi}{\partial x} &= 0,
\end{aligned}
$$

with φ pollutant concentration.

All these equations are evolution problems of the form

$$\frac{\partial \boldsymbol{f}}{\partial t} + \boldsymbol{A}(\boldsymbol{f}) \cdot \nabla \boldsymbol{f} = \boldsymbol{S}(\boldsymbol{f})$$

with $\boldsymbol{f}$ the dependant function, $\boldsymbol{S}$ the source term (possibly a differential operator, e.g. diffusion), $\boldsymbol{A}$ a matrix.

Hyperbolic problems

Hyperbolic problems share a number of properties

- they describe systems in which *information* spreads at finite velocity
- this information can be conserved (when the source term is zero) or altered (nonzero source term)
- solutions can be discontinuous
- smooth boundary and initial conditions can give rise to discontinuous solutions after a finite time

Characteristic equation for one-dimensional problems

Let us first consider the following advection equation with $n = 1$ space variable and without source term:

$$\partial_t u(x,\ t) + a(u)\partial_x u(x,\ t) = 0,$$

subject to one boundary condition of the form:

$$u(x,\ 0) = u_0(x) \text{ at } t = 0.$$

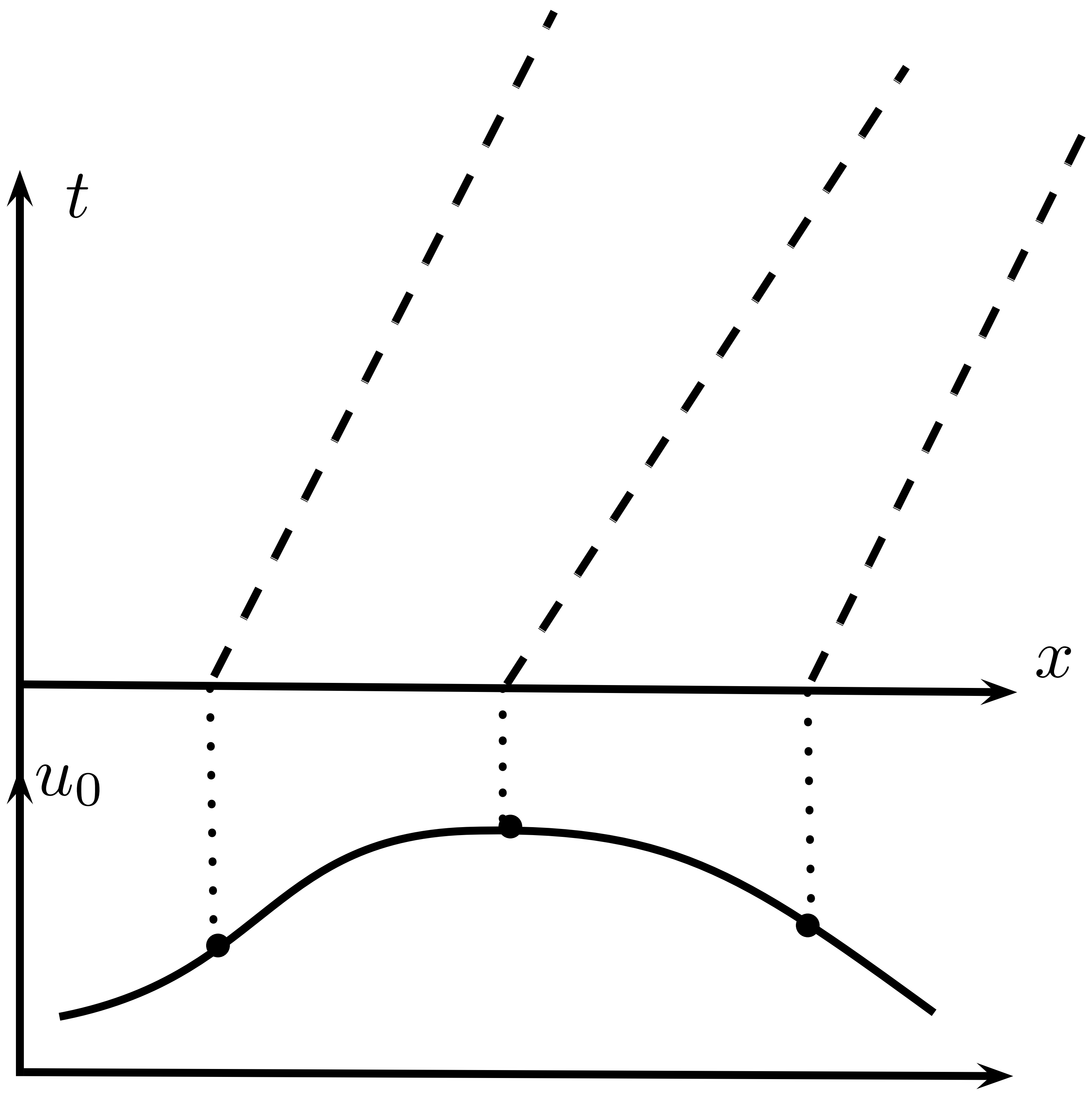
Note the this PDE is equivalent to

$$\partial_t u(x,\ t) + \partial_x f[u(x,\ t)] = 0,$$

with $a = f'(u)$ when f is C^1 continuous.

A characteristic curve is a curve $x = x_c(t)$ along which the partial differential equation $\partial_f U + a\partial_x U = 0$ is equivalent to an ordinary differential equation.

Characteristic equation



Characteristic curves

Consider a solution $u(x,\ t)$ of the differential system. Along the curve $\mathcal{C}$ of equation $x = x_c(t)$ we have: $u(x,\ t) = u(x_c(t),\ t)$ and the rate change is:

$$\frac{\mathrm{d}u(x_c(t),\ t)}{\mathrm{d}t} = \frac{\partial u(x,\ t)}{\partial t} + \frac{\mathrm{d}x_c}{\mathrm{d}t}\frac{\partial u(x,\ t)}{\partial x}.$$

Suppose now that the curve $\mathcal{C}$ satisfies the equation $\mathrm{d}x_c/\mathrm{d}t = a(u)$:

$$\frac{\mathrm{d}u(x,\ t)}{\mathrm{d}t} = \frac{\partial u(x,\ t)}{\partial t} + a\frac{\partial u(x,\ t)}{\partial x} = 0.$$

Characteristic equation

Any convection equation can be cast in a characteristic form:

$$\boxed{\frac{\partial}{\partial t}u(x,\ t) + a(u)\frac{\partial}{\partial x}u(x,\ t) = 0 \Leftrightarrow \frac{\mathrm{d}u(x,\ t)}{\mathrm{d}t} = 0 \text{ along straight lines } \mathcal{C}\text{: } \frac{\mathrm{d}x}{\mathrm{d}t} = a(u).}$$

Since $\mathrm{d}u(x,\ t)/\mathrm{d}t = 0$ along $x_c(t)$, this means that $u(x,\ t)$ is conserved along this curve. Since u is constant $a(u)$ is also constant, so the curves $\mathcal{C}$ are straight lines. This holds true for linear and nonlinear systems.

If the source term is non zero, this does not change the final equation (except for the right-hand term), but u is no longer conserved.

Characteristic equation

When this equation is subject to an initial condition, the characteristic equation can be easily solved. As u is constant along the characteristic line, we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(u) \Rightarrow x - x_0 = a(u)(t - t_0),$$

with the initial condition $t_0 = 0$, $u(x,\ t) = u_0(x)$. We then infer

$$x - x_0 = a(u_0(x_0))t$$

is the equation for the (straight) characteristic line emanating from point x_0. Furthermore, $t \geq 0$ $u(x,\ t) = u_0(x_0)$ since u is conserved. Since we have: $x_0 = x - a(u_0(x_0))t$, we then deduce:

$$\boxed{u(x,\ t) = u_0(x - a(u_0(x_0))t).}$$

Shock formation

Consider the convective nonlinear equation:

$$\frac{\partial}{\partial t}u(x,\, t) + \frac{\partial}{\partial x}f[u(x,\, t)] = 0,$$

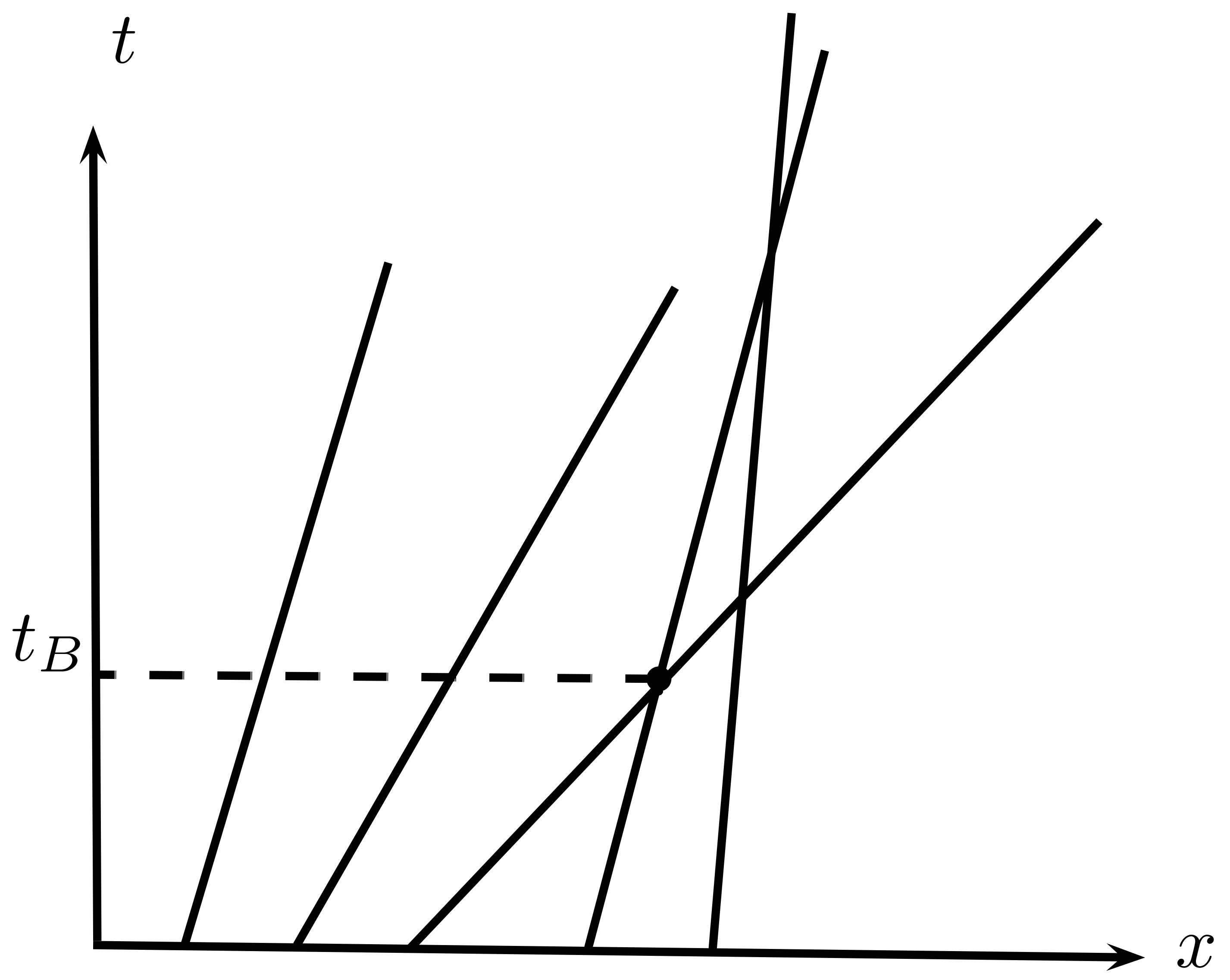
with initial condition $u(x,\, 0) = u_0(x)$ and f a given function of u. This equation can be solved simply by the method of characteristics.

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \text{ along curves } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda(u),$$

where $\lambda(u) = f'(u)$. We deduce that u is constant along the characteristic curves. So $\mathrm{d}x/\mathrm{d}t = \lambda(u) = c$, with c a constant that can be determined using the initial condition: the characteristics are straight lines with slopes $\lambda(u_0(x_0))$ depending on the initial condition:

$$x = x_0 + \lambda(u_0(x_0))t.$$

Shock formation



Characteristic curves and shock formation

Since u is constant along a characteristic curve, we find:

$$u(x,\ t) = u_0(x_0) = u_0\left(x - \lambda(u_0(x_0))t\right)$$

The characteristic lines can intersect in some cases, especially when the characteristic velocity decreases: $\lambda'(u) < 0$. What happens then? When two characteristic curves intersect, this means that potentially, u takes two different values, which is not possible for a continuous solution. The solution becomes discontinuous: a shock is formed.

Shock formation

When two characteristic curves interest, the differential u_x becomes infinite (since u takes two values at the same time). We can write u_x as follows

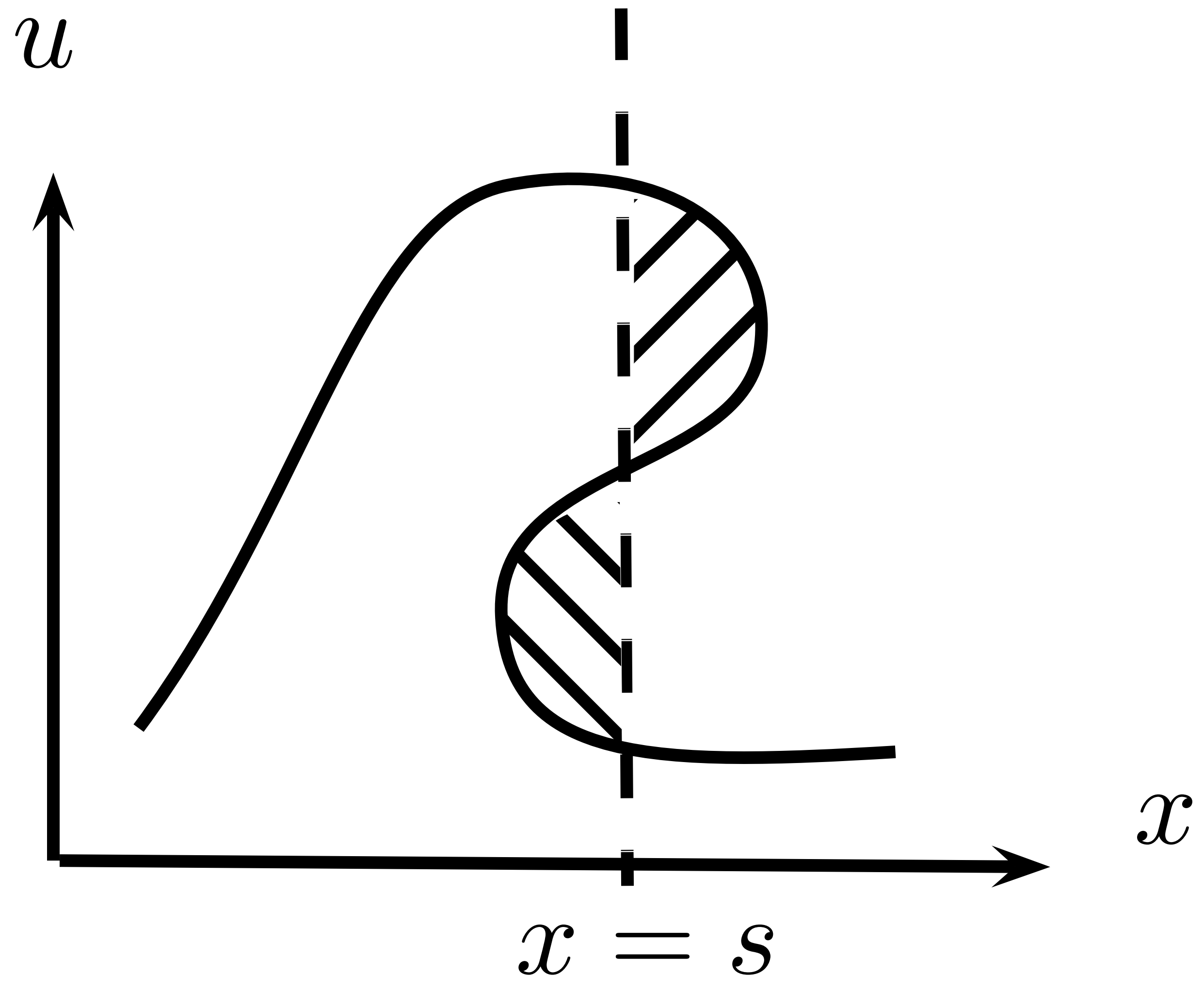
$$u_x = u_0'(x_0)\frac{\partial x_0}{\partial x} = u_0'(x_0)\frac{1}{1+\lambda'(u_0(x_0))u'(x_0)t} = \frac{u_0'(x_0)}{1+\partial_x\lambda(x_0)t},$$

where we used the relation: $\lambda'(u_0(x_0))u'(x_0) = \partial_u\lambda\partial_x u = \partial_x\lambda$. The differential u_x tends to infinity when the denominator tends to 0, i.e. at time: $t_b = -1/\lambda'(x_0)$. At the crossing point, u changes its value very fast: a shock is formed. The $s = s(t)$ line in the $x - t$ plane is the shock locus. A necessary condition for shock occurrence is then $t_b > 0$:

$$\boxed{\lambda'(x_0) < 0.}$$

Therefore there is a slower speed characteristic.

Shock formation



Shock position

The characteristic curves that are causing the shock form an envelope curve whose implicit equation is given by:

$$x = x_0 + \lambda(u_0(x_0))t \text{ et } \lambda'(u_0(x_0)) + 1 = 0.$$

After the shock, the solution is multivalued, which is impossible from a physical standpoint. The multivalued part of the curve is then replaced with a discontinuity positioned so that the lobes of both sides are of equal area.

Shock formation: Rankine-Hugoniot equation

Generally, we do not attempt to calculate the envelope of characteristic curves, because there is a much simpler method to calculate the trajectory of the shock. Indeed, the original PDE can be cast in the integral form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_L}^{x_R} u(x,\ t)\mathrm{d}x = f(u(x_L,\ t)) - (u(x_R,\ t)),$$

where x_L and x_R are abscissa of fixed point of a control volume. If the solution admits a discontinuity in $x = s(t)$ on the interval $[x_L,\ x_R]$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_L}^{x_R} u(x,\ t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{x_L}^{s} u(x,\ t)\mathrm{d}x + \int_{s}^{x_R} u(x,\ t)\mathrm{d}x\right),$$

That is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_L}^{x_R} u(x,\ t)\mathrm{d}x = \int_{x_L}^{s} \frac{\partial}{\partial t}u(x,\ t)\mathrm{d}x + \int_{s}^{x_R} \frac{\partial}{\partial t}u(x,\ t)\mathrm{d}x + \dot{s}u(x_L, t) - \dot{s}u(x_R, t).$$

Shock formation: Rankine-Hugoniot equation

Taking the limit $x_R \to s$ and $x_L \to s$, we deduce:

$$\dot{s}[\![u]\!] = [\![f(u)]\!],$$

where

$$[\![u]\!] = u^+ - u^- = \lim_{x \to s, x > s} u - \lim_{x \to s, x < s} u,$$

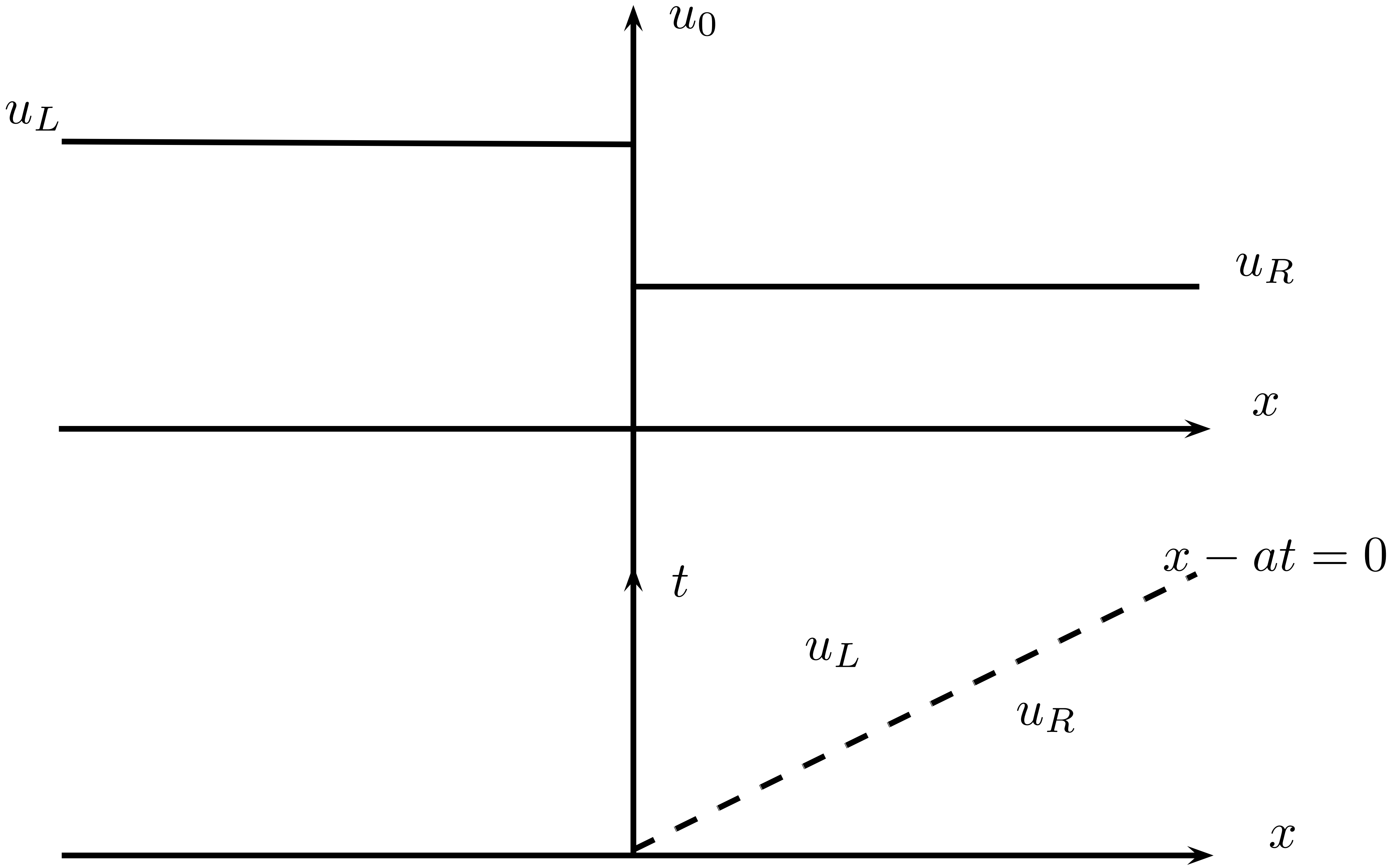
The $+$ and $-$ signs are used to describe what is happening on the right and left, respectively, of the discontinuity at $x = s(t)$.

In conclusion, we must have on both sides of $x = s(t)$:

$$\boxed{\dot{s}[\![u]\!] = [\![f(u)]\!]}$$

This is the *Rankine-Hugoniot* equation.

Riemann problem



Riemann problem for the linear case

We call *Riemann problem* an initial-value problem of the following form:

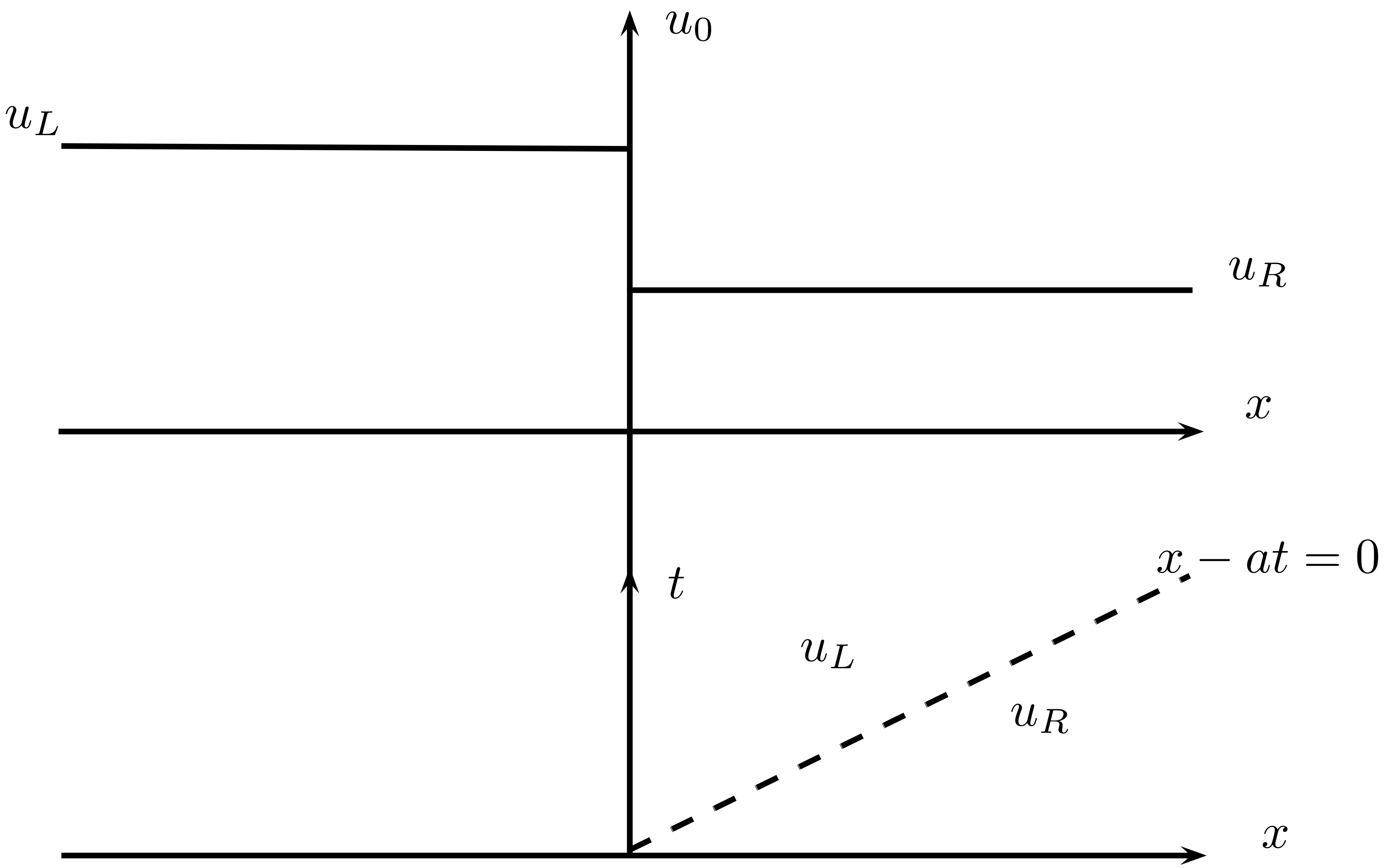
$$\partial_t u + \partial_x[f(u)] = 0,$$

$$u(x,\ 0) = u_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases}$$

with u_L et u_R two constants.

This problem describes how an initially piecewise constant function u, with a discontinuity in $x = 0$ changes over time. This problem is fundamental to solving theoretical and numerical problems.

Riemann problem: linear case



Riemann problem for the linear case

Let us consider the linear case $f(u) = au$, with a a constant. The solution is straightforward:

$$u(x,\ t) = u_0(x - at) = \begin{cases} u_L & \text{if } x - at < 0, \\ u_R & \text{if } x - at > 0. \end{cases}$$

The discontinuity propagates with a speed a.

Riemann problem: nonlinear case

In the general case (where $f'' \neq 0$), the Riemann problem is an initial-value problem of the following form:

$$\partial_t u + \partial_x[f(u)] = 0,$$

$$u(x,\ 0) = u_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0. \end{cases}$$

with u_L and u_R two constants. Assume that $f'' > 0$ (the case of a non-convex flow will not be treated here). We will show that there are two possible solutions:

- a solution called *rarefaction wave* (or simple wave), which is continuous;
- a discontinuous solution which represents the spread of the initial discontinuity (*shock*).

Riemann problem: nonlinear case

Rarefaction wave. The PDE is invariant under the transformation $x \to \lambda x$ and $t \to \lambda t$. A general solution can be sought in the form $U(\xi)$ with $\xi = x/t$. Substituting this general form into the partial differential equation, we obtain an ordinary differential equation of the form:

$$\left(f'(U(\xi)) - \xi\right) U' = 0.$$

There are two types of solution to this equation:

- *rarefaction wave*: $\left(f'(U(\xi)) - \xi\right) = 0$. If $f'' > 0$, then $f'(u_R) > f'(u_L)$; equation $f'(U) = \xi$ admits a single solution when $f'(u_R) > \xi > f'(u_L)$. In this case, u_L is connected to u_R through *a rarefaction wave*: $\xi = f'(U(\xi))$. Inverting f', we find out the desired solution

$$u(x,\ t) = f'^{(-1)}(\xi)$$

Riemann problem: nonlinear case

- *constant state*: $U'(\xi) = 0$. This is the trivial solution $u(x,\ t) = \mathrm{cst}$. This solution does not satisfy the initial problem.

The solution is thus a rarefaction wave. It reads

$$u(x,\ t) = \begin{cases} u_L & \text{if } \dfrac{x}{t} \leq f'(u_L), \\ f'^{(-1)}(\xi) & \text{si } f'(u_L) \leq \dfrac{x}{t} \leq f'(u_R) \\ u_R & \text{if } \dfrac{x}{t} \geq f'(u_R). \end{cases}$$

Riemann problem: nonlinear case

Shock wave

Weak solutions (discontinuous) to the hyperbolic differential equation may exist. Assuming a discontinuity along a line $x = s(t) = \dot{s}t$, we get: $[\![f(u)]\!] = \dot{s}[\![u]\!]$. The solution is then:

$$u(x,\ t) = \begin{cases} u_L \text{ if } x < \dot{s}t, \\ u_R \text{ if } x > \dot{s}t. \end{cases}$$

Then a shock wave forms, with its velocity $\dot{s}$ given by:

$$\boxed{\dot{s} = \frac{f(u_L) - f(u_R)}{u_L - u_R}.}$$

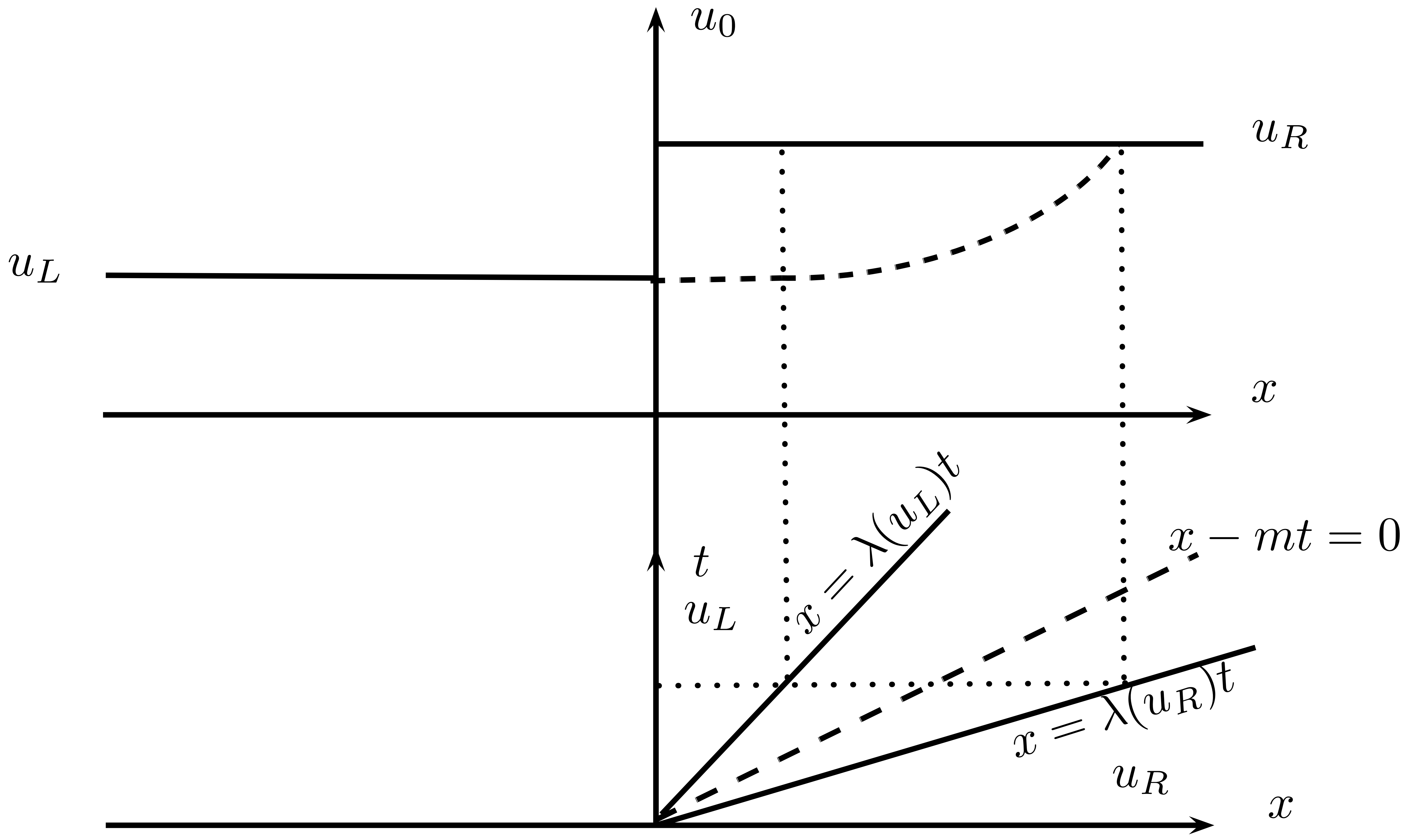
Riemann problem: nonlinear case

Selection of the physical solution

Two cases are to be considered (remember that $f'' > 0$). We call $\lambda(u) = f'(u)$ the *characteristic velocity* (see section below), which is the slope of the characteristic curve (straight line) of the problem.

- 1st case: $u_R > u_L$. Since $f'' > 0$, then $\lambda(u_R) > \lambda(u_L)$. At initial time $t = 0$, the characteristic lines form a fan. Equation $\xi = f'(U(\xi))$ admits a solution over the interval $\lambda(u_R) > \xi > \lambda(u_L)$;
- 2nd case: $u_R < u_L$. Characteristic lines intersect as of $t = 0$. The shock propagates at rate $\lambda(u_R) < \dot{s} < \lambda(u_L)$. This last condition is called *Lax condition*; it allows to determining whether the shock velocity is physically admissible.

Riemann problem: nonlinear case



Riemann problem: nonlinear case

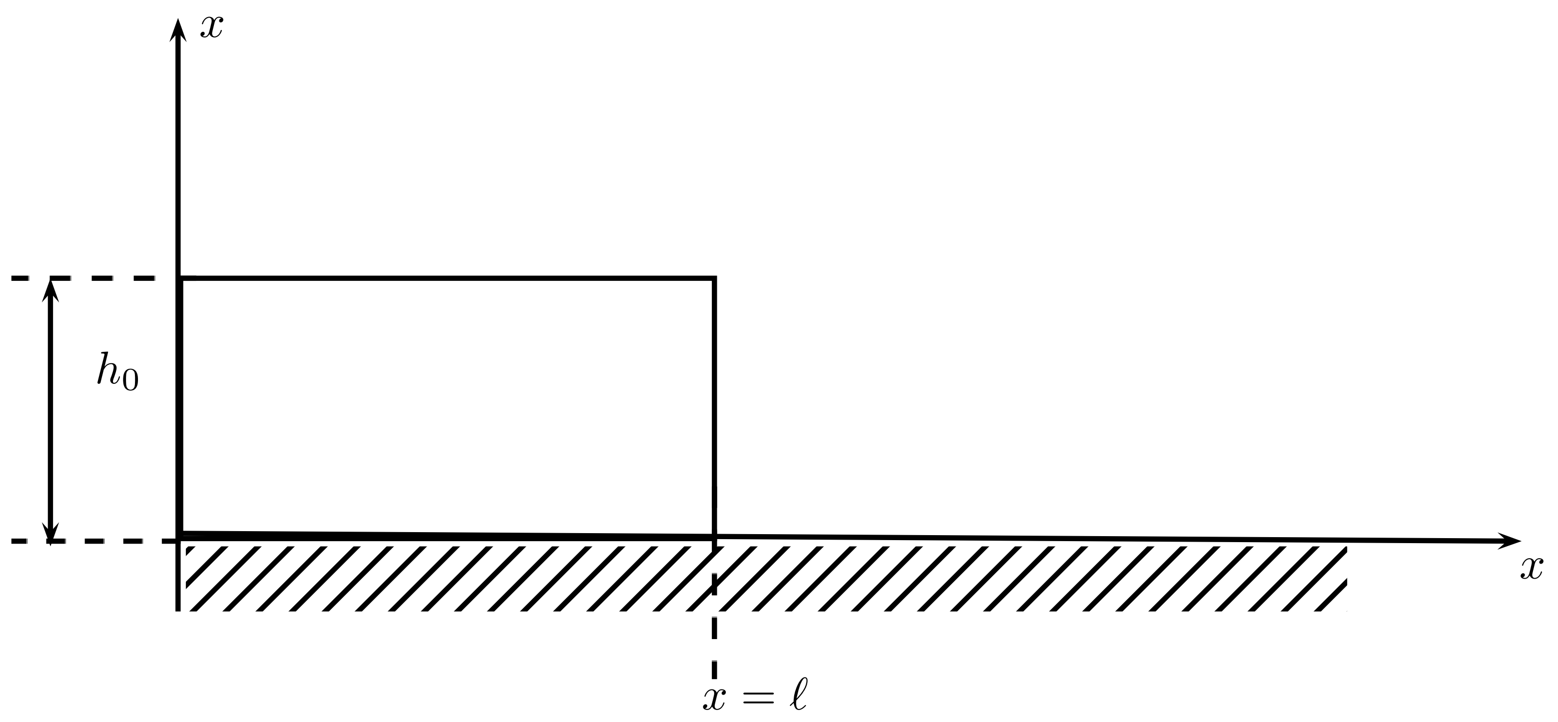
Non-convex flux

For some applications, the flux is not convex. An example is given by the equation of Buckley-Leverett, reflecting changes in water concentration ϕ in a pressure-driven flow of oil in a porous medium:

$$\phi_t + f(\phi)_x = 0,$$

with $f(\phi) = \phi^2(\phi^2 + a(1-\phi)^2)^{-1}$ and a a parameter $(0 < a < 1)$. This fonction has an inflexion point. Contrary to the convex case, for which the solution involves shock and rarefaction waves, the solution is here made up of shocks and *compound wave* resulting from the superimposition of one shock wave and one rarefaction wave.

Exercise 1



Solve Huppert's equation, which describes fluid motion over an inclined plane in the low Reynolds-number limit:

$$\frac{\partial h}{\partial t} + \frac{\rho g h^2 \sin\theta}{\mu}\frac{\partial h}{\partial x} = 0.$$

The solution must also satisfy the mass conservation equation

$$\int h(x,t)\mathrm{d}x = V_0$$

where V is the initial volume $V_0 = \ell h_0$

Generalization to higher dimensions: terminology

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Terminology

We study evolution equations in the form:

$$\boldsymbol{U}_t + \boldsymbol{A}(\boldsymbol{U})\boldsymbol{U}_x + \boldsymbol{B} = \boldsymbol{0},$$

with $\boldsymbol{A}$ an $n \times n$ matrix. $\boldsymbol{B}$ is a vector of dimension n called the *source*. The system is homogeneous if $\boldsymbol{B} = \boldsymbol{0}$. It is a conservative form when

$$\boldsymbol{U}_t + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{0},$$

with $\boldsymbol{A}(\boldsymbol{U}) = \partial \boldsymbol{F}/\partial \boldsymbol{U}$.

The eigenvalues λ_i of $\boldsymbol{A}$ represent the speed(s) at which information propagates. They are the zeros of the polynomial $\det(\boldsymbol{A} - \lambda \boldsymbol{1}) = 0$. The system is *hyperbolic* if $\boldsymbol{A}$ has n real eigenvalues.

Generalization to higher dimensions: terminology

If a function satisfies an evolution equation:

$$u_t + [f(u)]_x = 0,$$

then we can create an infinity of equivalent PDEs: $[g(u)]_t + [h(u)]_x = 0$ provided that g and h are such that $h' = g'f'$. As long as the function $u(x, t)$ is continuously differentiable, there is no problem, but for weak solutions (exhibiting a discontinuity), then the equations are no longer equivalent. We must use the original physical equation (usually expressing conservation of mass, momentum or energy).

Left and right eigenvectors

Take the particular case $n = 2$ for illustration. The matrix $\boldsymbol{A}$ has two real eigenvalues λ_1 and λ_2 together with *left eigenvectors* $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$:

$$\boldsymbol{v}_i \cdot \boldsymbol{A} = \lambda_i \boldsymbol{v}_i.$$

It also has two right eigenvectors $\boldsymbol{w}_1$ et $\boldsymbol{w}_2$:

$$\boldsymbol{A} \cdot \boldsymbol{w}_i = \lambda_i \boldsymbol{w}_i.$$

Let us assume that $\boldsymbol{A}$ has the following entries

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

Left and right eigenvectors

Then we get

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ \dfrac{d-a+\sqrt{\Delta}}{2c} \end{pmatrix}, \ \boldsymbol{w}_1 = \begin{pmatrix} \dfrac{a-d+\sqrt{\Delta}}{2c} \\ 1 \end{pmatrix}, \text{ associated with } \lambda_1 = \frac{a+d+\sqrt{\Delta}}{2},$$

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ \dfrac{d-a-\sqrt{\Delta}}{2c} \end{pmatrix}, \ \boldsymbol{w}_2 = \begin{pmatrix} \dfrac{a-d-\sqrt{\Delta}}{2c} \\ 1 \end{pmatrix}, \text{ associated with } \lambda_2 = \frac{a+d-\sqrt{\Delta}}{2},$$

avec $\Delta = (a-d)^2 + 4bc$. Note that

$$\boldsymbol{v}_1 \cdot \boldsymbol{w}_2 = 0, \text{ and } \boldsymbol{v}_2 \cdot \boldsymbol{w}_1 = 0.$$

Diagonalization

Linear system: When the eigenvectors are constant

$$\boldsymbol{v}_i \cdot \boldsymbol{U}_t + \boldsymbol{v}_i \cdot \boldsymbol{A}(\boldsymbol{U})\boldsymbol{U}_x + \boldsymbol{v}_i \cdot \boldsymbol{B} = \boldsymbol{0}.$$

thus:

$$\boldsymbol{v}_i \cdot \boldsymbol{U}_t + \lambda_i \boldsymbol{v}_i \cdot \boldsymbol{U}_x + \boldsymbol{v}_i \cdot \boldsymbol{B} = \boldsymbol{0}.$$

We pose $r_i = \boldsymbol{v}_i \cdot \boldsymbol{U}$ and obtain

$$\boldsymbol{r}_t + \boldsymbol{\Lambda} \cdot \boldsymbol{r}_x + \boldsymbol{r} \cdot \boldsymbol{B} = \boldsymbol{0}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1,\ \lambda_2\}$. The system is now made of independent PDEs

$$\frac{\mathrm{d}\boldsymbol{r}_1}{\mathrm{d}t} + \boldsymbol{r}_1 \cdot \boldsymbol{B} = \boldsymbol{0} \text{ along } x = x_{c,1}(t),\ \frac{\mathrm{d}x_{c,1}(t)}{\mathrm{d}t} = \lambda_1,$$
$$\frac{\mathrm{d}\boldsymbol{r}_2}{\mathrm{d}t} + \boldsymbol{r}_2 \cdot \boldsymbol{B} = \boldsymbol{0} \text{ along } x = x_{c,2}(t),\ \frac{\mathrm{d}x_{c,2}(t)}{\mathrm{d}t} = \lambda_2,$$

Diagonalization: nonlinear system

Nonlinear system: We seek new variables $\boldsymbol{r} = \{r_1,\ r_2\}$ such that:

$$\boldsymbol{v}_1 \cdot \mathrm{d}\boldsymbol{U} = \mu_1 \mathrm{d}r_1,$$

$$\boldsymbol{v}_2 \cdot \mathrm{d}\boldsymbol{U} = \mu_2 \mathrm{d}r_2,$$

where μ_i are integrating factors such that $\mathrm{d}r_i$ are exact differential. We have:

$$\mu_1 \mathrm{d}r_1 = \mu_1 \left(\frac{\partial r_1}{\partial U_1} \mathrm{d}U_1 + \frac{\partial r_1}{\partial U_2} \mathrm{d}U_2 \right) = v_{11} \mathrm{d}U_1 + v_{12} \mathrm{d}U_2.$$

Identifying the various terms leads to:

$$\frac{\partial r_1}{\partial U_1} = \frac{v_{11}}{\mu_1},$$

and

$$\frac{\partial r_1}{\partial U_2} = \frac{v_{12}}{\mu_1}.$$

Diagonalization: nonlinear system

By taking the ratio of the two equations above, we get:

$$\frac{\partial r_1}{\partial U_1} = \frac{v_{11}}{v_{12}} \frac{\partial r_1}{\partial U_2},$$

The Schwartz theorem states that $\partial_{xy} f = \partial_{yx} f$ and so from $\mathrm{d}u(x, y) = a\mathrm{d}x + b\mathrm{d}y$, we deduce that $\partial_y a = \partial_x b$. Here this gives us the relation

$$\frac{\partial}{\partial U_1} \frac{v_{12}}{\mu_1} = \frac{\partial}{\partial U_2} \frac{v_{11}}{\mu_1}.$$

The integrating factor can also be deduced from $\partial r_1/\partial U_2 = 1/\mu_1$ when the entries of $\boldsymbol{v}_1$ are properly selected such that $v_{11} = 1$. Note that

$$\frac{\partial r_1}{\partial U_1} = \frac{v_{11}}{v_{12}} \frac{\partial r_1}{\partial U_2} \Rightarrow w_{21} \partial r_1/\partial U_1 + w_{22} \partial r_1/\partial U_2 = 0 \Rightarrow \boldsymbol{w}_2 \cdot \nabla r_1 = 0$$

Definition: r_1 is said to be a 2-invariant of the system.

Diagonalization: nonlinear system

The characteristic equation associated with the equation above is

$$\frac{\mathrm{d}U_1}{v_{12}} = \frac{\mathrm{d}U_2}{v_{11}} = \frac{\mathrm{d}r_1}{0},$$

which leads to an integral. The first equation of the differential system is equivalent to:

$$\boldsymbol{v}_1 \cdot \left.\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}\right|_{x=X_1(t)} + \boldsymbol{v}_1 \cdot \boldsymbol{B} = 0,$$

where $x = X_1(t)$ satisfies $\mathrm{d}X_1/\mathrm{d}t = \lambda_1$. This is the *1-characteristic curve*:

$$\mu_1 \left.\frac{\mathrm{d}r_1}{\mathrm{d}t}\right|_{x=X_1(t)} + \boldsymbol{v}_1 \cdot \boldsymbol{B} = 0.$$

Similarly for r_2:

$$\mu_2 \left.\frac{\mathrm{d}r_2}{\mathrm{d}t}\right|_{x=X_2(t)} + \boldsymbol{v}_2 \cdot \boldsymbol{B} = 0.$$

Diagonalization: nonlinear system

In a matrix form:

$$\left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_{\boldsymbol{r}=\boldsymbol{X}(t)} + \boldsymbol{S}(\boldsymbol{r},\ \boldsymbol{B}) = \boldsymbol{0},$$

along two characteristic curves $\boldsymbol{r} = \boldsymbol{X}(t)$ such that $\mathrm{d}\boldsymbol{X}(t)/\mathrm{d}t = (\lambda_1,\ \lambda_2)$; $\boldsymbol{S}$ is the source term whose entries are $\mu_i S_i = \boldsymbol{v}_i \cdot \boldsymbol{B}$. The new variables $\boldsymbol{r}$ are called the *Riemann variables*. For $\boldsymbol{B} = 0$, they are constant along the characteristic curves and thus they are called *Riemann invariants*.

Exercise 2

Consider the Saint-Venant equations:

$$\partial_t h + \partial_x(uh) = 0, \tag{1}$$

$$\partial_t u + u\partial_x u + \partial_x h = 0, \tag{2}$$

Determine the Riemann invariants and plot the characteristic curve for the dam-break problem

– initial velocity $-\infty < x < \infty$ $u(x, 0) = 0$

– initial depth $x < 0$ $h(x, 0) = h_0$

$x > 0$ $h(x, 0) = 0$

The Riemann problem: linear systems

Consider the following linear hyperbolic problem:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{U}}{\partial x} = 0,$$

where $\boldsymbol{A}$ is an $n \times n$ matrix with n distinct real eigenvalues. We thus have $\boldsymbol{A} = \boldsymbol{R} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{R}^{-1}$, with $\boldsymbol{R}$ the matrix associated with the change of coordinates (the columns are the right eigenvectors of $\boldsymbol{A}$) and $\boldsymbol{\Lambda}$ a diagonal matrix whose entries are λ_i. Making use of the change of variables $\boldsymbol{W} = \boldsymbol{R}^{-1} \cdot \boldsymbol{U}$ leads to

$$\frac{\partial \boldsymbol{W}}{\partial t} + \boldsymbol{\Lambda} \cdot \frac{\partial \boldsymbol{W}}{\partial x} = 0.$$

This is a system of independent linear hyperbolic PDEs: $\partial_t w_i + \lambda_i \partial_x w_i = 0$, whose solution takes the form $w_i = \omega_i(x - \lambda_i t)$.

The Riemann problem: linear systems

The inverse change of variables leads to e $\boldsymbol{U} = \boldsymbol{R} \cdot \boldsymbol{W}$

$$\boldsymbol{U} = \sum_{i=1}^{n} w_i(x,\ t)\boldsymbol{r}_i,$$

where $\boldsymbol{r}_i$ is a right eigenvector $\boldsymbol{A}$ associate to λ_i and w_i is ith entry of $\boldsymbol{W}$. The solution results from the superimposition of n waves travelling at speed λ_i; these waves are independent, do not change form (this form is given by the initial condition $\omega_i(x,\ 0)\boldsymbol{r}_i$). When all but one elementary waves are constant ($\partial_x \omega_i(x,\ 0) = 0$), then the resulting wave is called a j-*simple wave*

$$\boldsymbol{U} = \omega_j(x - \lambda_j t)\boldsymbol{r}_j + \sum_{i=1,\ i\neq j}^{n} w_i(x,\ t)\boldsymbol{r}_i,$$

Information propagates along the j-characteristic curve (all others w_i are constant).

The Riemann problem: linear systems

The Riemann problem takes the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{U}}{\partial x} = 0,$$

with

$$\boldsymbol{U}(x,\ 0) = \boldsymbol{U}_0(x) = \begin{cases} \boldsymbol{U}_\ell & \text{if } x < 0, \\ \boldsymbol{U}_r & \text{if } x > 0. \end{cases}$$

We now expand $\boldsymbol{U}_\ell$ et $\boldsymbol{U}_r$ in the eigenvector basis $\boldsymbol{r}_i$

$$\boldsymbol{U}_\ell = \sum_{i=1}^{n} w_i^{(\ell)} \boldsymbol{r}_i \text{ et } \boldsymbol{U}_r = \sum_{i=1}^{n} w_i^{(r)} \boldsymbol{r}_i,$$

with $\boldsymbol{w}_\ell = w_i^{(\ell)}$ et $\boldsymbol{w}_r = w_i^{(r)}$ vectors with constant entries.

The Riemann problem: linear systems

The Riemann problem involves n scalar problems

$$w_i(x,\ 0) = \begin{cases} w^{(\ell)} & \text{if } x < 0, \\ w^{(r)} & \text{if } x > 0. \end{cases}$$

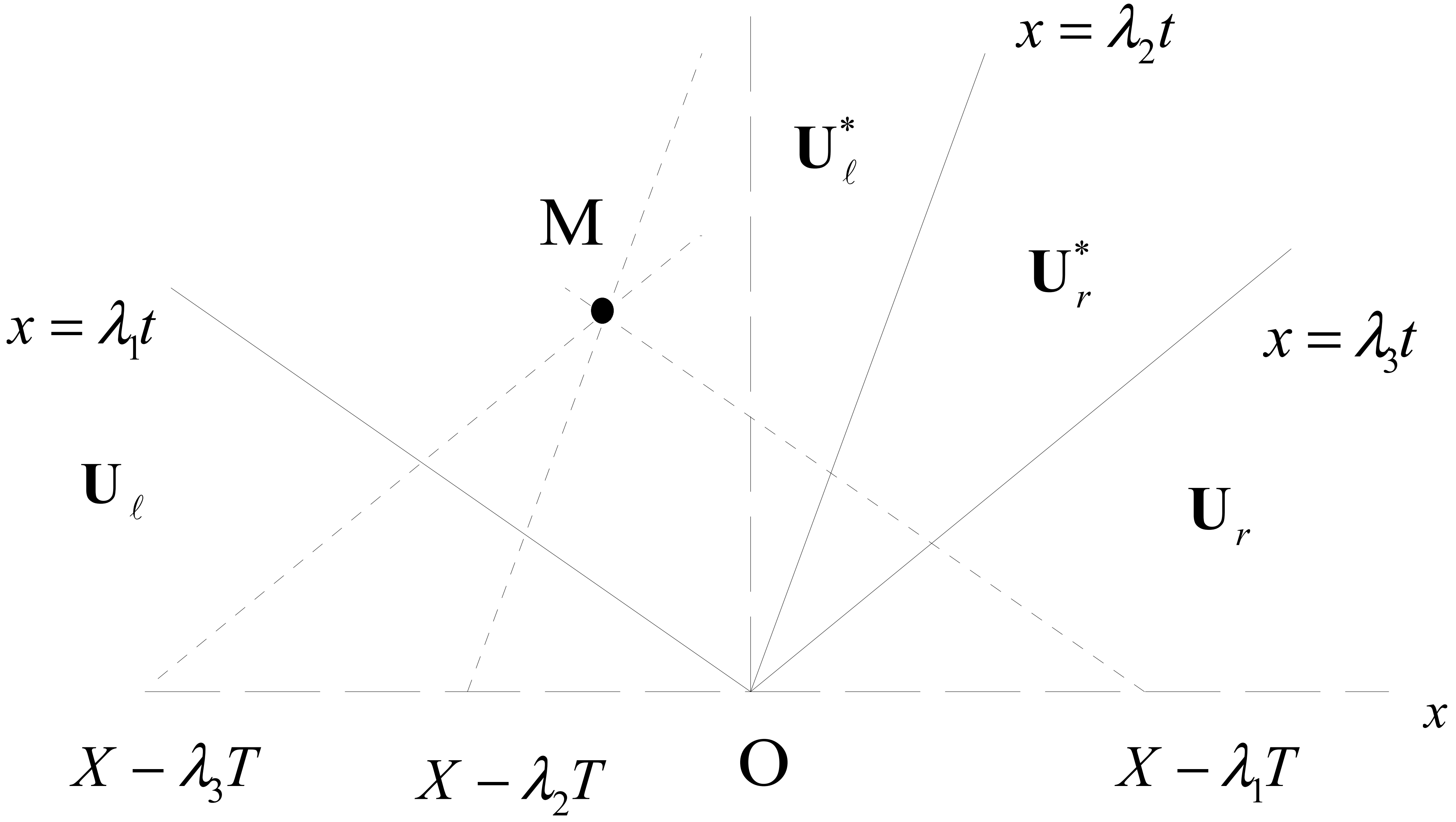
The solution to these advection equations is

$$w_i(x,\ t) = \begin{cases} w_i^{(\ell)} & \text{if } x - \lambda_i t < 0, \\ w_i^{(r)} & \text{if } x - \lambda_i t > 0. \end{cases}$$

We call $I(x,\ t)$ the largest index i such that $x - \lambda_i t$. The solution reads

$$\boldsymbol{U}(x,\ t) = \sum_{i=1}^{I} w_i^{(r)} \boldsymbol{r}_i + \sum_{i=I+1}^{n} w_i^{(\ell)} \boldsymbol{r}_i.$$

The Riemann problem: linear systems



Consider the case $n = 3$. The solution in the $x - t$ space breaks down into "wedges" where $\boldsymbol{U}$ is constant and separated by characteristic curves $x = \lambda_i t$. At any point M, we can determine the value taken by $\boldsymbol{U}$ by plotting the characteristic curves issuing from M toward the x-axis.

Exercise 3

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

with initial data

$$u(x,0) = u_0(x) \text{ and } u_t(x,0) = u_1(x)$$

Solve the equation.

The Riemann problem: nonlinear systems

Consider the following linear hyperbolic problem:

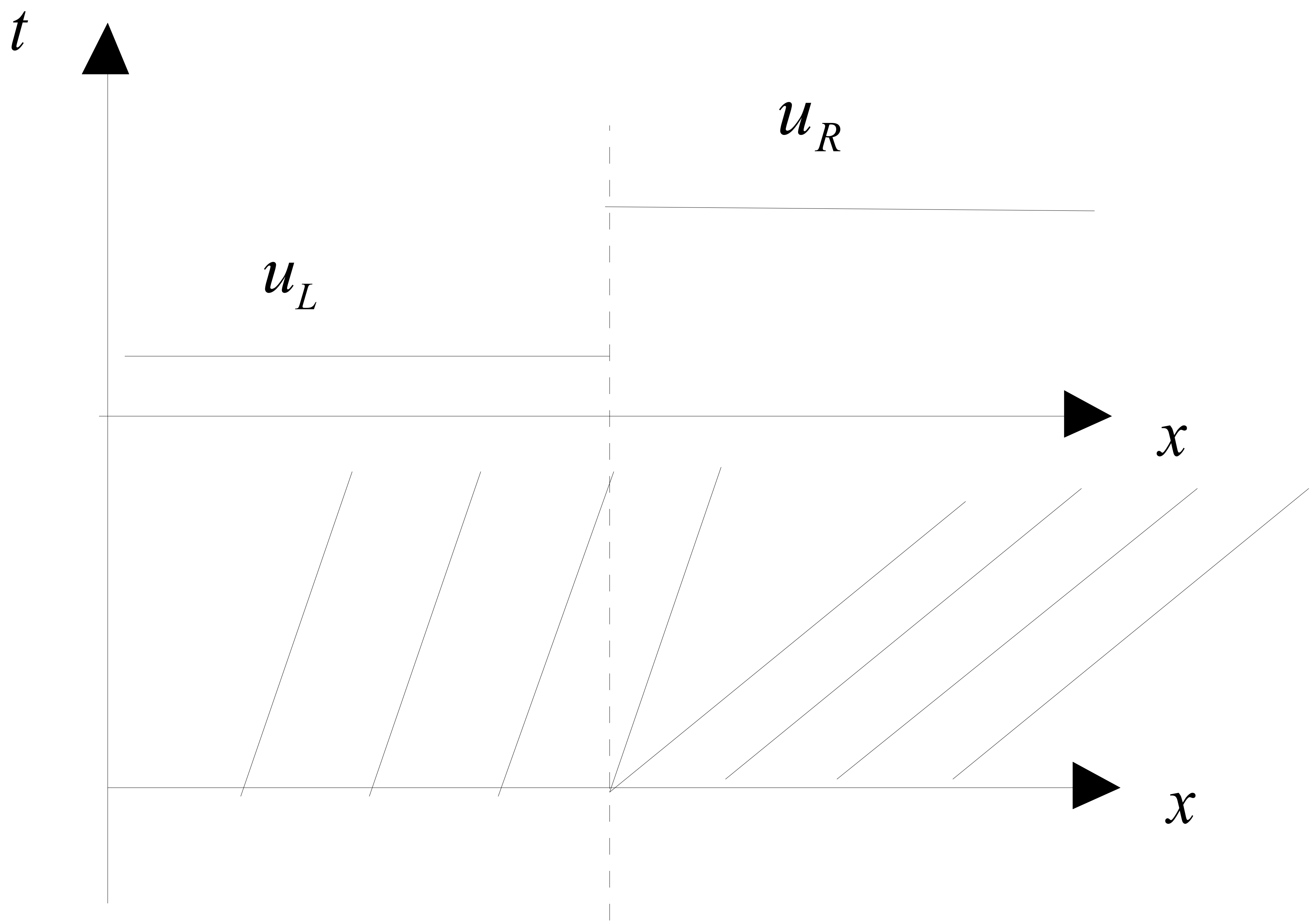
$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{U}}{\partial x} = 0,$$

where $\boldsymbol{A} = \nabla \boldsymbol{F}$. This equation is invariant to the stretching group $(x, t) \to (\lambda x, \lambda t)$. We seek solutions in the form $\boldsymbol{U}(x,\ t) = \boldsymbol{W}(\xi, u_L, u_R)$, with $\xi = x/t$

$$-\xi \frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}\xi} + \nabla \boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}\xi} = 0$$

- $\boldsymbol{W}'(\xi) = 0$, this is the constant state;
- $\boldsymbol{W}'(\xi)$ is a right eigenvector of $\nabla \boldsymbol{F}$ associate to ξ for all values taken by ξ. The curve $\boldsymbol{W}(\xi)$ is tangent to the right eigenvector $\boldsymbol{w}$.

The Riemann problem: nonlinear systems



Generalizing the concept seen for 1D hyperbolic equations, we define a rarefaction wave as a simple wave function of $\xi = x/t$

$$\boxed{\boldsymbol{u}(\xi) = \begin{cases} \boldsymbol{u}_L \text{ si } x/t \leq \xi_1, \\ \boldsymbol{W}(\xi, u_L, u_R) \text{ si } \xi_1 \leq x/t \leq \xi_2, \\ \boldsymbol{u}_R \text{ si } x/t \geq \xi_2. \end{cases}}$$

where $\boldsymbol{u}_R$ and $\boldsymbol{u}_L$ must satisfy

$\lambda_k(\boldsymbol{u}_L) < \lambda_k(\boldsymbol{u}_R)$

The Riemann problem: nonlinear systems

From the original PDE

$$-\xi\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}\xi}+\nabla\boldsymbol{F}\cdot\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}\xi}=0$$

we deduce that $\boldsymbol{W}'$ is a right eigenvector and that

$$\xi=\lambda_k(\boldsymbol{W}),$$

and on differentiating with respect to ξ, we get

$$1=\nabla_{\boldsymbol{u}}\lambda_k(\boldsymbol{W})\cdot\boldsymbol{W}'(\xi),$$

Since $\boldsymbol{W}'$ is a right eigenvector, $\boldsymbol{W}'(\xi)=\alpha\boldsymbol{w}_k$, thus $\alpha=[\nabla_{\boldsymbol{u}}\lambda_k(\boldsymbol{W})\cdot\boldsymbol{w}_k]^{-1}$. The function $\boldsymbol{W}$ is solution to the ODE

$$\boxed{\boldsymbol{W}'(\xi)=\frac{\boldsymbol{w}_k}{\nabla_{\boldsymbol{u}}\lambda_k(\boldsymbol{W})\cdot\boldsymbol{w}_k},}$$

The Riemann problem: nonlinear systems

Consider the following linear hyperbolic problem:

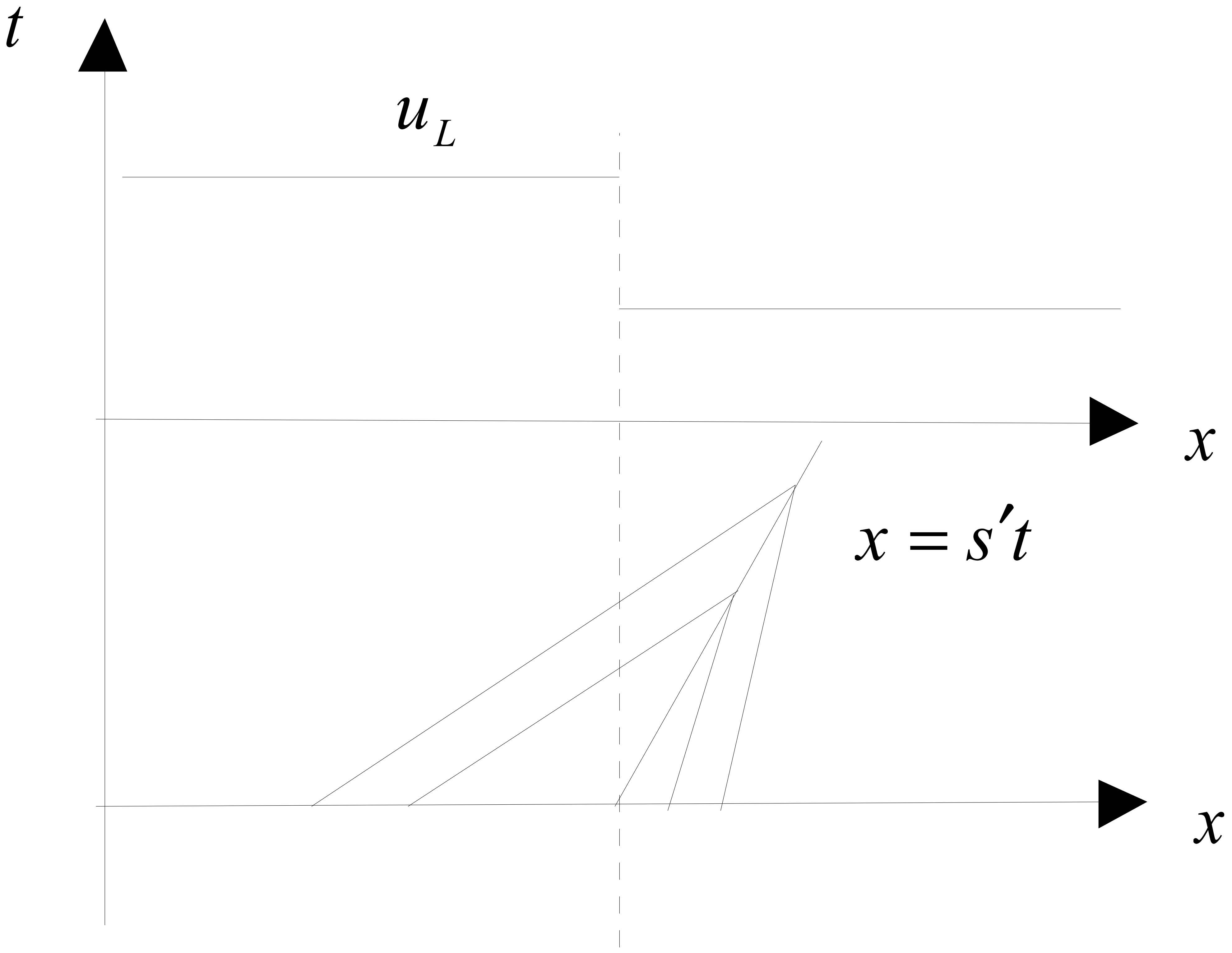
$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{U}}{\partial x} = 0,$$

where $\boldsymbol{A} = \nabla \boldsymbol{F}$. This equation is invariant to the stretching group $(x, t) \to (\lambda x, \lambda t)$. We seek solutions in the form $\boldsymbol{U}(x,\ t) = \boldsymbol{W}(\xi, u_L, u_R)$, with $\xi = x/t$

$$-\xi \frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}\xi} + \nabla \boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}\xi} = 0$$

- $\boldsymbol{W}'(\xi) = 0$, this is the constant state;
- $\boldsymbol{W}'(\xi)$ is a right eigenvector of $\nabla \boldsymbol{F}$ associate to ξ for all values taken by ξ. The curve $\boldsymbol{W}(\xi)$ is tangent to the right eigenvector $\boldsymbol{w}$.

The Riemann problem: nonlinear systems



A shock wave is a non-material surface $x = s(t)$ across which the solution is discontinuous $\dot{x} = s$. The Rankine-Hugoniot relation must hold

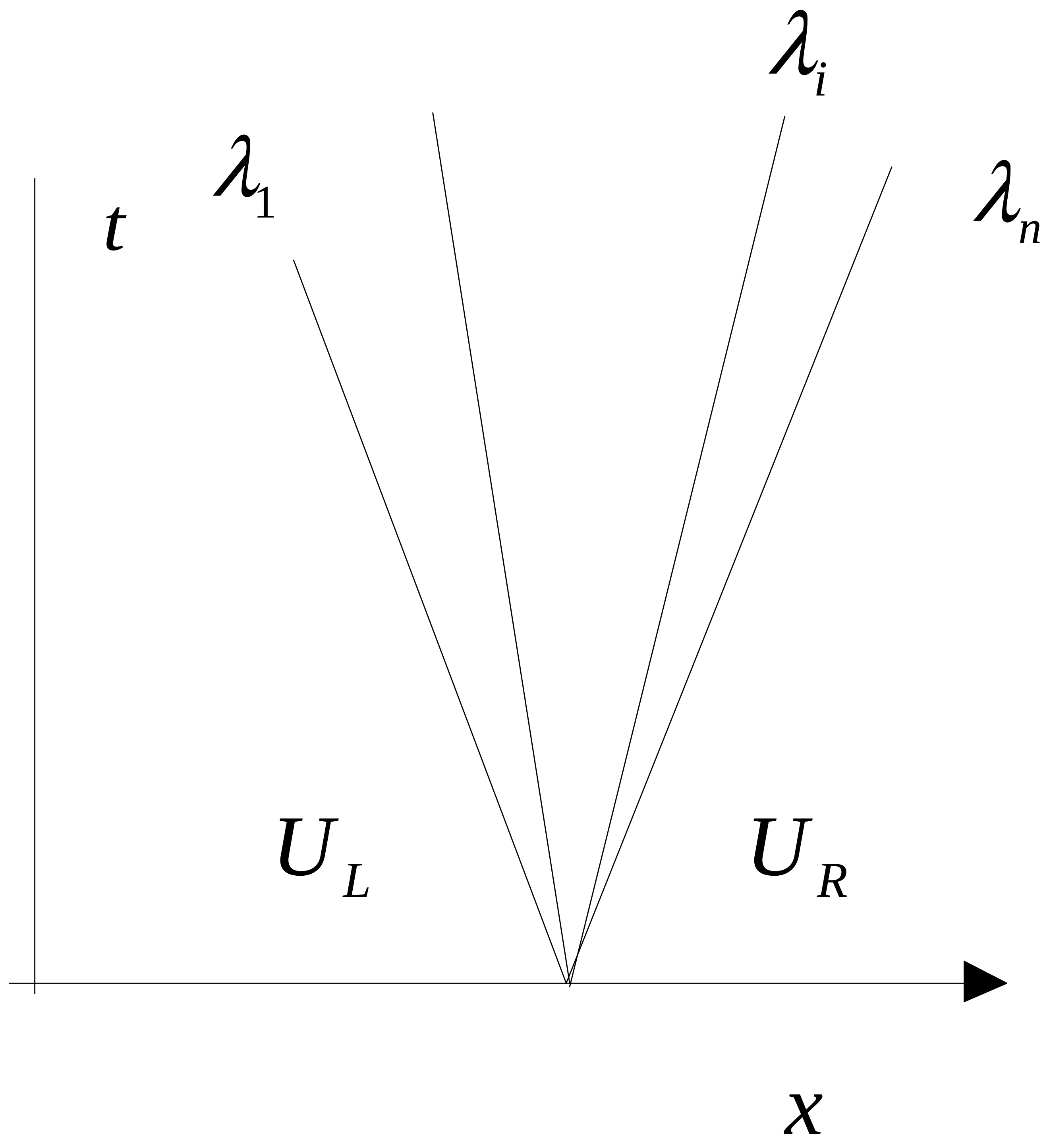
$$\boxed{\dot{s}(\boldsymbol{u}_L - \boldsymbol{u}_R) = f(\boldsymbol{u}_L) - f(\boldsymbol{u}_R),}$$

to which we add the *Lax entropy condition*

$$\boxed{\lambda_k(\boldsymbol{u}_L) > s > \lambda_k(\boldsymbol{u}_L),}$$

(jump in the kth field: we speak of a k-shock wave)

The Riemann problem: nonlinear systems



Summary

The solution to the Riemann problem:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} = 0,$$

subject to

$$\boldsymbol{U}(x,\ 0) = \boldsymbol{U}_0(x) = \begin{cases} \boldsymbol{U}_L & \text{si } x < 0, \\ \boldsymbol{U}_R & \text{si } x > 0. \end{cases}$$

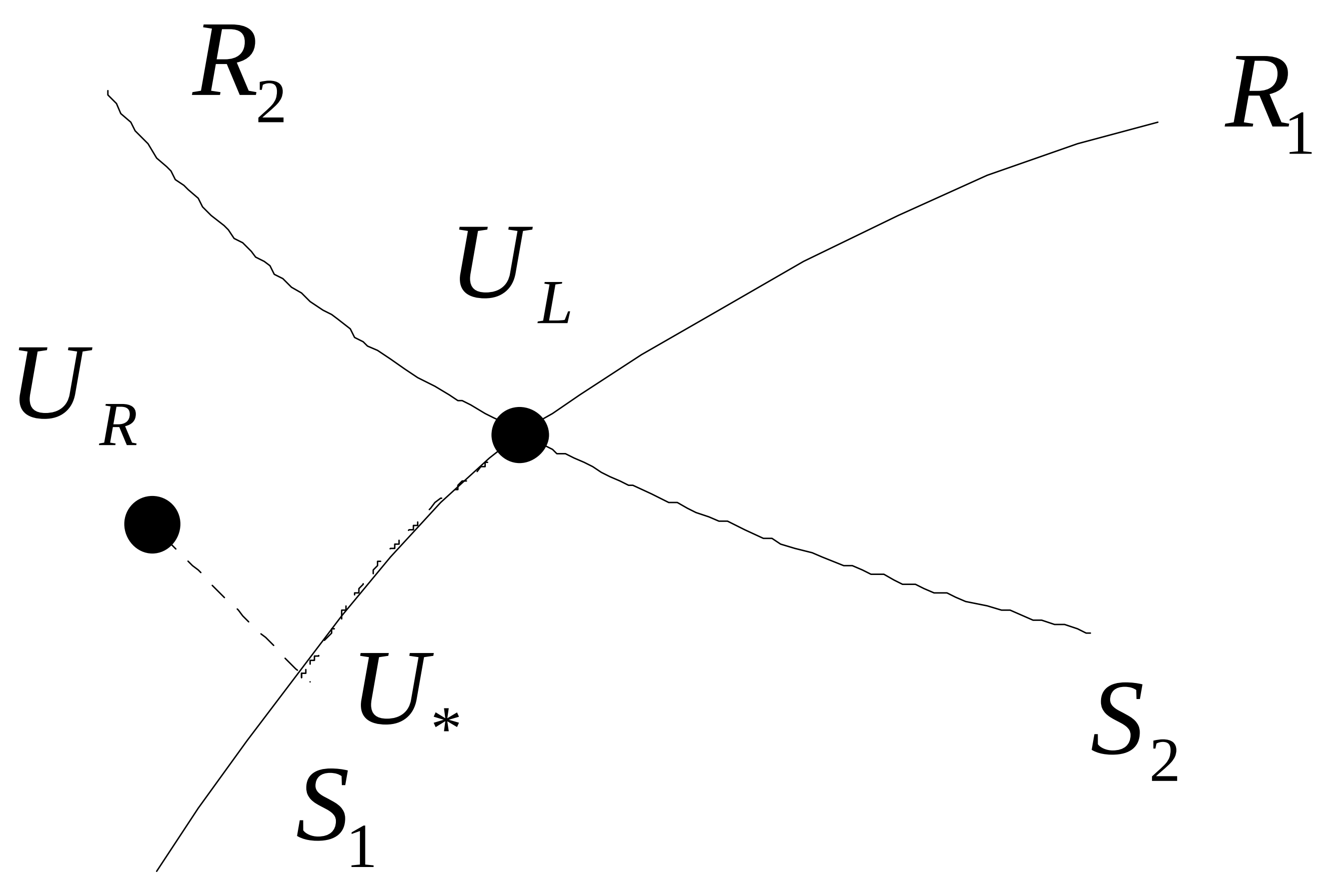
involves $n+1$ states separated by n waves related to each eigenvalue.

The Riemann problem: nonlinear systems

For linear systems, the eigenvalues define shock waves. For nonlinear systems, different types of waves are possible:

- *shock wave*: in this case, the Rankine-Hugoniot holds $s' \left[\boldsymbol{U}\right]_{x=s(t)} = \boldsymbol{F}(\boldsymbol{U}(x_L)) - \boldsymbol{F}(\boldsymbol{U}(x_R))$ along with the entropy condition $\lambda_i(\boldsymbol{U}_L) > s'_i > \lambda_i(\boldsymbol{U}_R)$
- *contact discontinuity* (when an eigenvalue is constant or such that $\nabla_{\boldsymbol{u}}\lambda_k \cdot \boldsymbol{w}_k = 0$): the Rankine-Hugoniot relation holds, with the condition $\lambda_i(\boldsymbol{U}_L) = \lambda_i(\boldsymbol{U}_R)$
- *rarefaction wave*: the characteristics fan out $\lambda_i(\boldsymbol{U}_L) < \lambda_i(\boldsymbol{U}_R)$, self-similar solutions.

The Riemann problem: nonlinear systems



Hugoniot locus

The solution to the Riemann problem:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} = 0,$$

subject to

$$\boldsymbol{U}(x,\ 0) = \boldsymbol{U}_0(x) = \begin{cases} \boldsymbol{U}_L & \text{si } x < 0, \\ \boldsymbol{U}_R & \text{si } x > 0. \end{cases}$$

involves $n+1$ states separated by n waves related to each eigenvalue.

Example: The Saint-Venant equations

Let us consider the Saint-Venant equations:

$$\partial_t h + \partial_x(uh) = 0,$$
$$\partial_t hu + \partial_x hu^2 + gh\partial_x h = 0.$$

We introduce the unknowns $\boldsymbol{U} = (h,\ hu)$, the flux function $\boldsymbol{F} = (hu,\ hu^2 + gh^2/2)$ and the matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} = \begin{pmatrix} 0 & 1 \\ gh - u^2 & 2u \end{pmatrix}.$$

The conservative form is:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{u}}{\partial x} = 0.$$

Example: The Saint-Venant equations

Eigenvalues and eigenvectors for the conservative formulation (with $c = \sqrt{gh}$)

	$i = 1$	$i = 2$
λ_i	$u - c$	$u + c$
$\boldsymbol{w}_i$	$\{\frac{1}{u-c}, 1\}$	$\{\frac{1}{u+c}, 1\}$
$\boldsymbol{w}_i \cdot \nabla\lambda_i$	$\frac{3c}{2(c-u)}$	$\frac{3c}{2(c+u)}$

Example: The Saint-Venant equations

If we take (h, u) as variables, then the system is put in a nonconservative, but some solutions are easier to work out. With $\boldsymbol{U} = (h,\ u)$, $\boldsymbol{F} = (hu,\ hu^2 + gh^2/2)$ and matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} = \begin{pmatrix} u & h \\ g & u \end{pmatrix},$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{u}}{\partial x} = 0.$$

Example: The Saint-Venant equations

Eigenvalues and eigenvectors for the nonconservative formulation (with $c = \sqrt{gh}$)

		$i = 1$	$i = 2$
eigenvalues	λ_i	$u - c$	$u + c$
right eigenvectors	$\boldsymbol{w}_i$	$\left\{-\frac{c}{g}, \ 1\right\}$	$\left\{\frac{c}{g}, \ 1\right\}$
left eigenvectors	$\boldsymbol{v}_i$	$\left\{-\frac{c}{h}, \ 1\right\}$	$\left\{\frac{c}{h}, \ 1\right\}$
Riemann invariants	r_i	$u + 2c$	$u - 2c$

Example: The Saint-Venant equations

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Shock conditions

$$\sigma[\![h]\!] = [\![hu]\!],$$
$$\sigma[\![hu]\!] = [\![hu^2 + gh^2/2]\!],$$

with σ the shock velocity. In a frame related to the shock wave, then $v = u - \sigma$ and

$$h_1 v_1 = h_2 v_2,$$
$$h_1 v_1^2 + g h_1^2/2 = h_2 v_2^2 + g h_2^2/2.$$

There are two families

- 1-shock: $\sigma < u_L - c_L$ et $u_R - c_R < \sigma < u_R + c_R$. $v_L > v_R$: the flux goes from left to right when $v_L > 0$;
- 2-shock: $\sigma > u_R + c_R$ et $u_L - c_L < \sigma < u_L + c_L$. $v_R > v_L$: the flux goes from right to left when $v_L > 0$.

Example: The Saint-Venant equations

Let us determine the Hugoniot locus, i.e., the points $(h_2 \;\; v_2)$ connected to $(h_1 \;\; v_1)$ by a 1- or 2-shock wave

$$\sigma = \frac{h_2 v_2 - h_1 v_1}{h_2 - h_1},$$

$$\frac{(h_2 u_2 - h_1 u_1)^2}{h_2 - h_1} = h_2 u_2^2 + \frac{g h_2^2}{2} - h_1 u_1^2 - \frac{g h_1^2}{2},$$

This gives us the shock speed and $u_2(h_2 | h_1 \;\; v_1)$:

$$u_2 = u_1 \mp (h_2 - h_1) \sqrt{\frac{g}{2} \frac{h_1 + h_2}{h_1 h_2}},$$

$$\sigma = u_1 \mp \sqrt{\frac{g}{2}(h_1 + h_2)\frac{h_2}{h_1}}.$$

Example: The Saint-Venant equations

Rarefaction waves

We seek Riemann invariants r_k, defined as $\nabla_{\boldsymbol{u}} r_k \cdot \boldsymbol{w}_k = 0$. We work with the variables $(h,\ u)$. The first invariant is:

$$-c\frac{\partial r}{\partial h} + \lambda_1 \frac{\partial r}{\partial u} = 0,$$

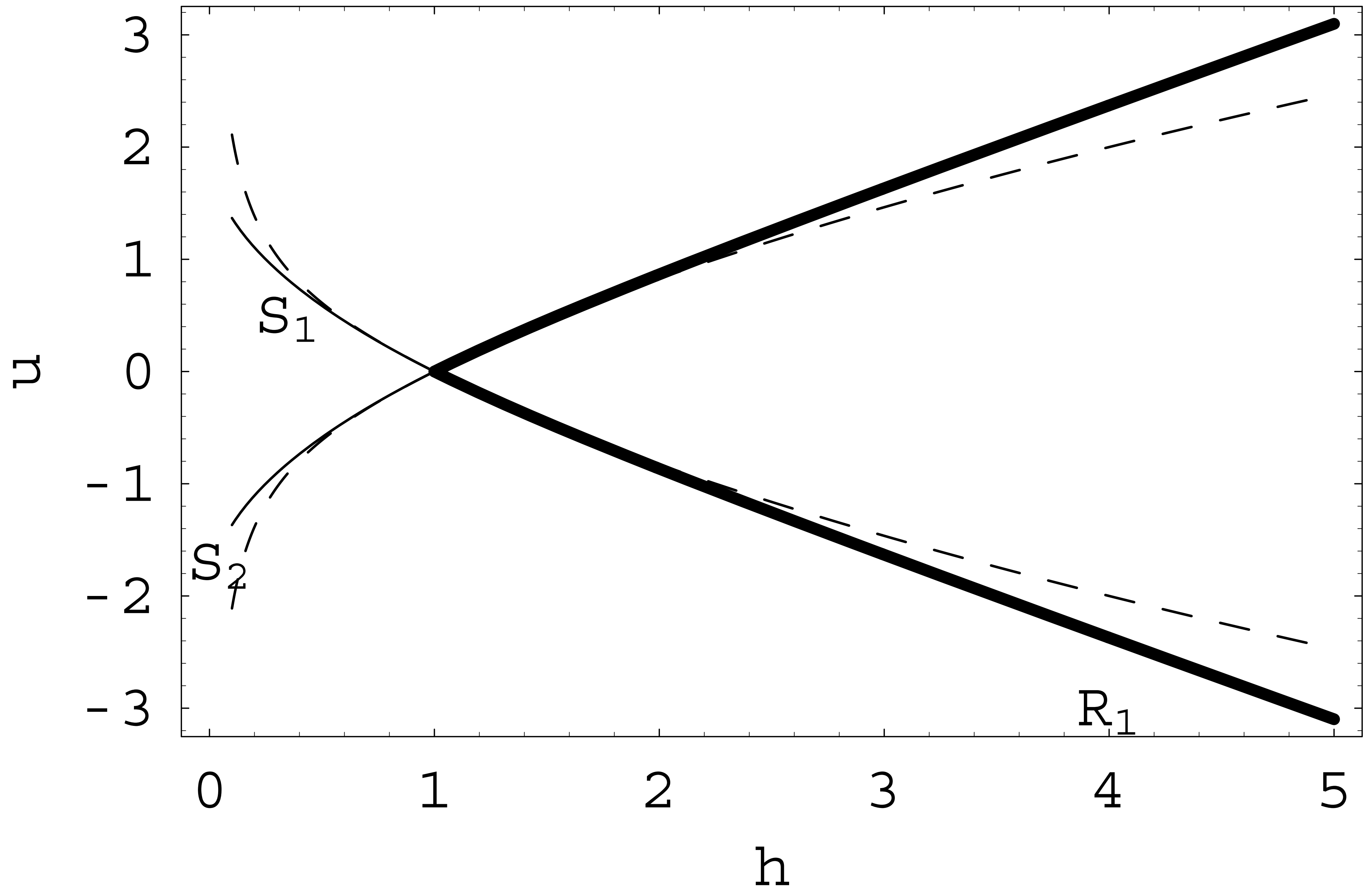
whose characteristic equations are

$$\frac{\mathrm{d}u}{g} = -\frac{\mathrm{d}h}{c}.$$

An integral is $u + 2c$. For the second invariant, we find $u - 2c$.

Along a 1-rarefaction wave, we have: $u_2 + 2\sqrt{gh_2} = u_1 + 2\sqrt{gh_1}$ and the invariant $r_1 = u + 2c$ is constant along any characteristic curve associated with the eigenvalue $\lambda_1 = u - c$ (when these fan out, r_1 is in the cone formed by the characteristics).

Example: The Saint-Venant equations



Show and rarefaction waves in the (h, u) space. Arbitrarily the curves are issuing from $(h, u) = (1, 0)$

Returning to the variables $(h,\ q = hu)$, we deduce

- Along a 1-rarefaction wave, we get: $q_2/h_2 + 2\sqrt{gh_2} = q_1/h_1 + 2\sqrt{gh_1}$;
- Along a 2-rarefaction wave, we get: $q_2/h_2 - 2\sqrt{gh_2} = q_1/h_1 - 2\sqrt{gh_1}$.

Example: The Saint-Venant equations

Working out the solution to the Riemann problem

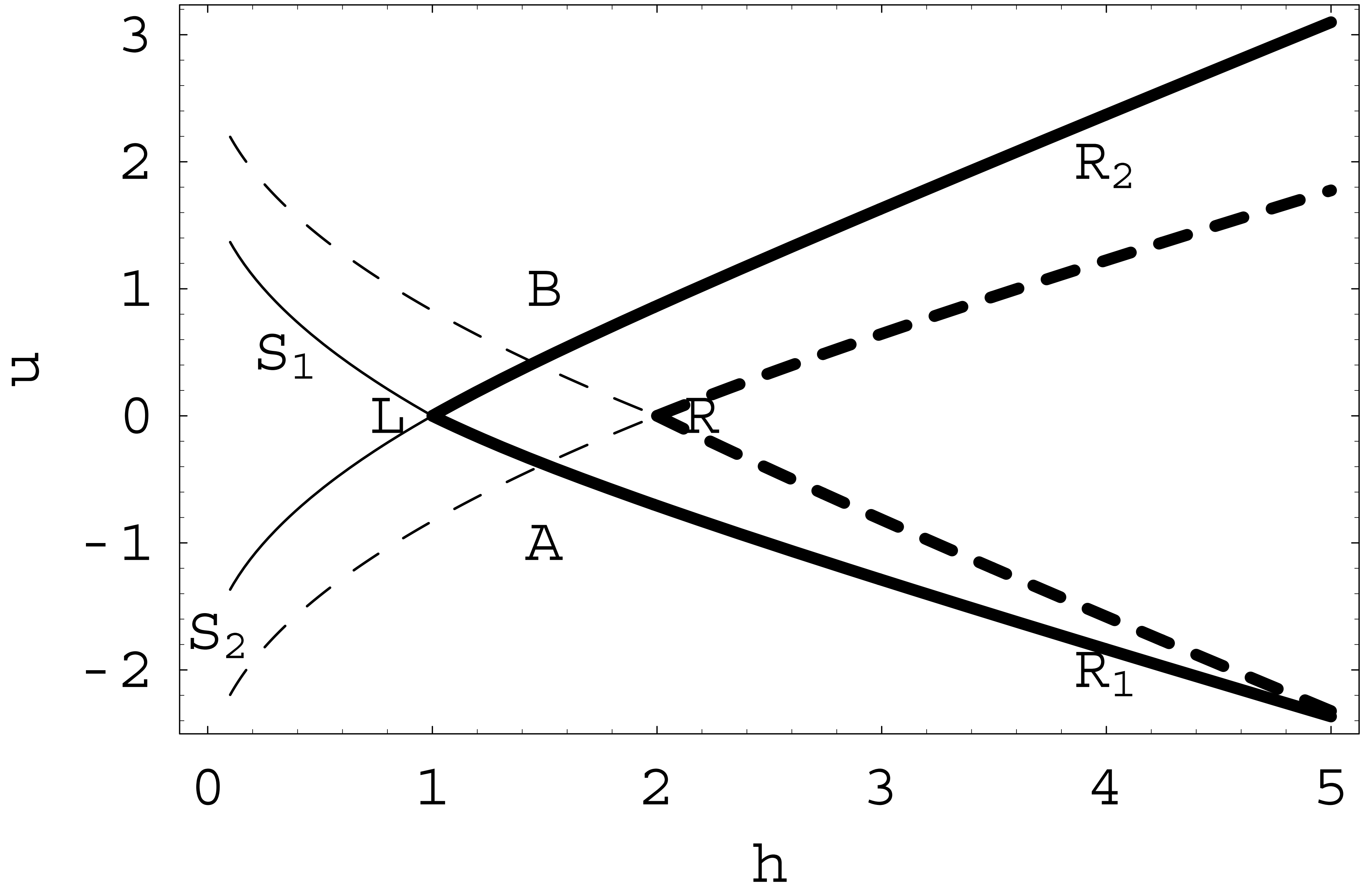
The construction method consists of introducing an intermediate state u_*. The state (h_*, u_*) can be connected to a left state (h_L, u_L) through a 1-wave

$$u_* = \begin{cases} S_1(h_*|\ h_L,\ u_L) = u_L + 2\sqrt{gh_L} - 2\sqrt{gh_*} & \text{if } h_* < h_L \text{ 1-rarefaction wave} \\ R_1(h_*|\ h_L,\ u_L) = u_L - (h_* - h_L)\sqrt{g\dfrac{h_* + h_L}{2h_*h_L}} & \text{if } h_* > h_L \text{ 1-shock wave} \end{cases}$$

It can be connected to a right state (h_R, u_R) through a 2-wave

$$u_* = \begin{cases} S_2(h_*|\ h_R,\ u_R) = u_R - 2\sqrt{gh_R} + 2\sqrt{gh_*} & \text{if } h_* < h_R \text{ 2-rarefaction wave} \\ R_2(h_*|\ h_R,\ u_R) = u_R + (h_* - h_R)\sqrt{g\dfrac{h_* + h_R}{2h_*h_R}} & \text{if } h_* > h_R \text{ 2-shock wave} \end{cases}$$

Example: The Saint-Venant equations



Solution to the Riemann problem for $(h_L, u_L) = (1, 0)$ et $(h_R, u_R) = (2, 0)$

We begin with 1-waves, then 2-waves as information on the left gauche is primarily conveyed by the smallest eigenvalue, then the others.

Note that tangents to the curves R_1 et S_1 are the same. Note also that an intermediate state is possible only if:

$$u_R - u_L < 2(\sqrt{gh_R} + \sqrt{gh_L}).$$

For $h_L = 0$ ($h_R = 0$, resp.), then the 1-shock wave (the 2-shock wave, resp.) is undefined.

Homework

Lacaze, L., A. Filella, and O. Thual, Steady and unsteady shear flows of a viscoplastic fluid in a cylindrical Couette cell, Journal of Non-Newtonian Fluid Mechanics, 220, 126-136, 2015.

Let us consider the (dimensionless) governing equations for a visco-elastoplastic material in a simple shear

$$\frac{\partial u}{\partial t} = 1 + \frac{\partial \tau}{\partial z},$$

$$\frac{\partial \tau}{\partial t} = \frac{\partial u}{\partial z} - F(\tau),$$

with $F(\tau) = \max\left(0, |\tau| - 1\right)^{1/n} \tau/|\tau|$. The boundary and initial conditions are $u = 0$ at $z = 0$, $\tau = 0$ at $z = 1$, and $\tau = u = 0$ at $t = 0$. Cast the system into its characteristic form. Write a numerical code to solve the resulting system.