Chapter 8: Hyperbolic partial differential equations

Similarity and Transport Phenomena in Fluid Dynamics

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Chapter 8: Hyperbolic partial differential equations

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Hyperbolic problems

Hyperbolic problems arise frequently in fluid mechanics (and continuum mechanics). For instance, in hydraulic engineering:

**Dimension 1:** nonlinear convection equation, for example the kinematic wave equation, which describes flood propagation in rivers

\[
\frac{\partial h}{\partial t} + K \sqrt{i} \frac{\partial h^{5/3}}{\partial x} = 0,
\]

with \( h \) flow depth, \( K \) Manning-Strickler coefficient, and \( i \) bed gradient;

**Dimension 2:** Saint-Venant equations (also called the shallow water equations)

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial h \bar{u}}{\partial x} &= 0, \\
\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} &= g \sin \theta - g \cos \theta \frac{\partial h}{\partial x} - \frac{\tau_p}{\partial h},
\end{align*}
\]

with \( \bar{u} \) flow-depth averaged velocity, \( h \) flow depth, \( \theta \) bed slope, and \( \tau_p \) bottom shear stress.
Hyperbolic problems

**Dimension 3:** Saint-Venant equations with advection of pollutant

\[
\frac{\partial h}{\partial t} + \frac{\partial \bar{u}}{\partial x} = 0, \\
\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} = g \sin \theta - g \cos \theta \frac{\partial h}{\partial x} - \frac{\tau_p}{\varrho h}, \\
\frac{\partial \phi}{\partial t} + \bar{u} \frac{\partial \phi}{\partial x} = 0,
\]

with \( \phi \) pollutant concentration.

All these equations are evolution problems of the form

\[
\frac{\partial f}{\partial t} + A(f) \cdot \nabla f = S(f)
\]

with \( f \) the dependant function, \( S \) the source term (possibly a differential operator, e.g. diffusion), \( A \) a matrix.
Hyperbolic problems share a number of properties

- they describe systems in which *information* spreads at finite velocity
- this information can be conserved (when the source term is zero) or altered (nonzero source term)
- solutions can be discontinuous
- smooth boundary and initial conditions can give rise to discontinuous solutions after a finite time
Let us first consider the following advection equation with $n = 1$ space variable and without source term:

$$\partial_t u(x, t) + a(u)\partial_x u(x, t) = 0,$$

subject to one boundary condition of the form:

$$u(x, 0) = u_0(x) \text{ at } t = 0.$$

Note the this PDE is equivalent to

$$\partial_t u(x, t) + \partial_x f[u(x, t)] = 0,$$

with $a = f'(u)$ when $f$ is $C^1$ continuous.

A characteristic curve is a curve $x = x_c(t)$ along which the partial differential equation $\partial_f U + a\partial_x U = 0$ is equivalent to an ordinary differential equation.
Consider a solution \( u(x, t) \) of the differential system. Along the curve \( C \) of equation \( x = x_c(t) \) we have: \( u(x, t) = u(x_c(t), t) \) and the rate change is:

\[
\frac{du(x_c(t), t)}{dt} = \frac{\partial u(x, t)}{\partial t} + \frac{dx_c}{dt} \frac{\partial u(x, t)}{\partial x}.
\]

Suppose now that the curve \( C \) satisfies the equation \( \frac{dx_c}{dt} = a(u) \):

\[
\frac{du(x, t)}{dt} = \frac{\partial u(x, t)}{\partial t} + a \frac{\partial u(x, t)}{\partial x} = 0.
\]
Any convection equation can be cast in a characteristic form:
\[
\frac{\partial}{\partial t} u(x, t) + a(u) \frac{\partial}{\partial x} u(x, t) = 0 \iff \frac{du(x, t)}{dt} = 0 \text{ along straight lines } C: \frac{dx}{dt} = a(u).
\]

Since \(du(x, t)/dt = 0\) along \(x_c(t)\), this means that \(u(x, t)\) is conserved along this curve. Since \(u\) is constant \(a(u)\) is also constant, so the curves \(C\) are straight lines. This holds true for linear and nonlinear systems.

If the source term is non zero, this does not change the final equation (except for the right-hand term), but \(u\) is no longer conserved.
When this equation is subject to an initial condition, the characteristic equation can be easily solved. As $u$ is constant along the characteristic line, we get

$$\frac{dx}{dt} = a(u) \Rightarrow x - x_0 = a(u)(t - t_0),$$

with the initial condition $t_0 = 0$, $u(x, t) = u_0(x)$. We then infer

$$x - x_0 = a(u_0(x_0))t$$

is the equation for the (straight) characteristic line emanating from point $x_0$. Furthermore, $t \geq 0$ $u(x, t) = u_0(x_0)$ since $u$ is conserved. Since we have:

$$x_0 = x - a(u_0(x_0))t,$$

we then deduce:

$$u(x, t) = u_0(x - a(u_0(x_0))t).$$
Consider the convective nonlinear equation:

\[
\frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} f[u(x, t)] = 0,
\]

with initial condition \( u(x, 0) = u_0(x) \) and \( f \) a given function of \( u \). This equation can be solved simply by the method of characteristics.

\[
\frac{du}{dt} = 0 \text{ along curves } \frac{dx}{dt} = \lambda(u),
\]

where \( \lambda(u) = f'(u) \). We deduce that \( u \) is constant along the characteristic curves. So \( \frac{dx}{dt} = \lambda(u) = c \), with \( c \) a constant that can be determined using the initial condition: the characteristics are straight lines with slopes \( \lambda(u_0(x_0)) \) depending on the initial condition:

\[
x = x_0 + \lambda(u_0(x_0)) t.
\]
Shock formation

Since $u$ is constant along a characteristic curve, we find:

$$u(x, t) = u_0(x_0) = u_0(x - \lambda(u_0(x_0))t)$$

The characteristic lines can intersect in some cases, especially when the characteristic velocity decreases: $\lambda'(u) < 0$. What happens then? When two characteristic curves intersect, this means that potentially, $u$ takes two different values, which is not possible for a continuous solution. The solution becomes discontinuous: a shock is formed.
Shock formation

When two characteristic curves interest, the differential $u_x$ becomes infinite (since $u$ takes two values at the same time). We can write $u_x$ as follows

$$u_x = u_0'(x_0) \frac{\partial x_0}{\partial x} = u_0'(x_0) \frac{1}{1 + \lambda'(u_0(x_0))u'(x_0)t} = \frac{u_0'(x_0)}{1 + \partial_x \lambda(x_0)t},$$

where we used the relation: $\lambda'(u_0(x_0))u'(x_0) = \partial_u \lambda \partial_x u = \partial_x \lambda$. The differential $u_x$ tends to infinity when the denominator tends to 0, i.e. at time: $t_b = -1/\lambda'(x_0)$. At the crossing point, $u$ changes its value very fast: a shock is formed. The $s = s(t)$ line in the $x - t$ plane is the shock locus. A necessary condition for shock occurrence is then $t_b > 0$:

$$\lambda'(x_0) < 0.$$

Therefore there is a slower speed characteristic.
The characteristic curves that are causing the shock form an envelope curve whose implicit equation is given by:

\[ x = x_0 + \lambda(u_0(x_0))t + \lambda'(u_0(x_0)) + 1 = 0. \]

After the shock, the solution is multivalued, which is impossible from a physical standpoint. The multivalued part of the curve is then replaced with a discontinuity positioned so that the lobes of both sides are of equal area.
Shock formation: Rankine-Hugoniot equation

Generally, we do not attempt to calculate the envelope of characteristic curves, because there is a much simpler method to calculate the trajectory of the shock. Indeed, the original PDE can be cast in the integral form:

$$\frac{d}{dt} \int_{x_L}^{x_R} u(x, t)dx = f(u(x_L, t)) - (u(x_R, t)),$$

where $x_L$ and $x_R$ are abscissa of fixed point of a control volume. If the solution admits a discontinuity in $x = s(t)$ on the interval $[x_L, x_R]$, then

$$\frac{d}{dt} \int_{x_L}^{x_R} u(x, t)dx = \frac{d}{dt} \left( \int_{x_L}^{s} u(x, t)dx + \int_{s}^{x_R} u(x, t)dx \right),$$

That is:

$$\frac{d}{dt} \int_{x_L}^{x_R} u(x, t)dx = \int_{x_L}^{s} \frac{\partial}{\partial t} u(x, t)dx + \int_{s}^{x_R} \frac{\partial}{\partial t} u(x, t)dx + \dot{s}u(x_L, t) - \dot{s}u(x_R, t).$$
Shock formation: Rankine-Hugoniot equation

Taking the limit $x_R \to s$ and $x_L \to s$, we deduce:

$$\dot{s}[u] = [f(u)],$$

where

$$[u] = u^+ - u^- = \lim_{x \to s, x > s} u - \lim_{x \to s, x < s} u,$$

The $+$ and $-$ signs are used to describe what is happening on the right and left, respectively, of the discontinuity at $x = s(t)$.

In conclusion, we must have on both sides of $x = s(t)$:

$$\dot{s}[u] = [f(u)]$$

This is the Rankine-Hugoniot equation.
Riemann problem

We call *Riemann problem* an initial-value problem of the following form:

$$\partial_t u + \partial_x [f(u)] = 0,$$

$$u(x, 0) = u_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases}$$

with $u_L$ et $u_R$ two constants.

This problem describes how an initially piecewise constant function $u$, with a discontinuity in $x = 0$ changes over time. This problem is fundamental to solving theoretical and numerical problems.
Let us consider the linear case $f(u) = au$, with $a$ a constant. The solution is straightforward:

$$u(x, t) = u_0(x - at) = \begin{cases} 
  u_L & \text{if } x - at < 0, \\
  u_R & \text{if } x - at > 0.
\end{cases}$$

The discontinuity propagates with a speed $a$. 
Riemann problem: nonlinear case

In the general case (where $f'' \neq 0$), the Riemann problem is an initial-value problem of the following form:

$$\partial_t u + \partial_x [f(u)] = 0,$$

$$u(x, 0) = u_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0. \end{cases}$$

with $u_L$ and $u_R$ two constants. Assume that $f'' > 0$ (the case of a non-convex flow will not be treated here). We will show that there are two possible solutions:

- a solution called rarefaction wave (or simple wave), which is continuous;
- a discontinuous solution which represents the spread of the initial discontinuity (shock).
Rarefaction wave. The PDE is invariant under the transformation $x \rightarrow \lambda x$ and $t \rightarrow \lambda t$. A general solution can be sought in the form $U(\xi)$ with $\xi = x/t$. Substituting this general form into the partial differential equation, we obtain an ordinary differential equation of the form:

$$(f'(U(\xi)) - \xi) U' = 0.$$ 

There are two types of solution to this equation:

- **rarefaction wave**: $(f'(U(\xi)) - \xi) = 0$. If $f'' > 0$, then $f'(u_R) > f'(u_L)$; equation $f'(U) = \xi$ admits a single solution when $f'(u_R) > \xi > f'(u_L)$. In this case, $u_L$ is connected to $u_R$ through a rarefaction wave: $\xi = f'(U(\xi))$. Inverting $f'$, we find out the desired solution

$$u(x, t) = f'^{-1}(\xi).$$
Riemann problem: nonlinear case

- **constant state**: $U'(\xi) = 0$. This is the trivial solution $u(x, t) = \text{cst}$. This solution does not satisfy the initial problem.

The solution is thus a rarefaction wave. It reads

$$u(x, t) = \begin{cases} u_L & \text{if } \frac{x}{t} \leq f'(u_L), \\ f''(-1)(\xi) & \text{if } f'(u_L) \leq \frac{x}{t} \leq f'(u_R), \\ u_R & \text{if } \frac{x}{t} \geq f'(u_R). \end{cases}$$
Riemann problem: nonlinear case

Shock wave

Weak solutions (discontinuous) to the hyperbolic differential equation may exist. Assuming a discontinuity along a line \( x = s(t) = \dot{s}t \), we get: \([f(u)] = \dot{s}[u]\). The solution is then:

\[
\begin{cases}
  u_L & \text{if } x < \dot{s}t, \\
  u_R & \text{if } x > \dot{s}t.
\end{cases}
\]

Then a shock wave forms, with its velocity \( \dot{s} \) given by:

\[
\dot{s} = \frac{f(u_L) - f(u_R)}{u_L - u_R}.
\]
Riemann problem: nonlinear case

Selection of the physical solution

Two cases are to be considered (remember that \( f'' > 0 \)). We call \( \lambda(u) = f'(u) \) the *characteristic velocity* (see section below), which is the slope of the characteristic curve (straight line) of the problem.

- **1st case:** \( u_R > u_L \). Since \( f'' > 0 \), then \( \lambda(u_R) > \lambda(u_L) \). At initial time \( t = 0 \), the characteristic lines form a fan. Equation \( \xi = f'(U(\xi)) \) admits a solution over the interval \( \lambda(u_R) > \xi > \lambda(u_L) \);

- **2nd case:** \( u_R < u_L \). Characteristic lines intersect as of \( t = 0 \). The shock propagates at rate \( \lambda(u_R) < \dot{s} < \lambda(u_L) \). This last condition is called *Lax condition*; it allows to determining whether the shock velocity is physically admissible.
Riemann problem: nonlinear case

\[ \begin{align*}
  x & = \lambda(u_L)t \\
  x & = \lambda(u_R)t \\
  x - mt & = 0
\end{align*} \]
Non-convex flux

For some applications, the flux is not convex. An example is given by the equation of Buckley-Leverett, reflecting changes in water concentration $\phi$ in a pressure-driven flow of oil in a porous medium:

$$\phi_t + f(\phi)_x = 0,$$

with $f(\phi) = \phi^2(\phi^2 + a(1 - \phi)^2)^{-1}$ and $a$ a parameter ($0 < a < 1$). This function has an inflexion point. Contrary to the convex case, for which the solution involves shock and rarefaction waves, the solution is here made up of shocks and compound wave resulting from the superimposition of one shock wave and one rarefaction wave.
Solve Huppert’s equation, which describes fluid motion over an inclined plane in the low Reynolds-number limit:

\[
\frac{\partial h}{\partial t} + \frac{\rho g h^2 \sin \theta}{\mu} \frac{\partial h}{\partial x} = 0.
\]

The solution must also satisfy the mass conservation equation

\[
\int h(x, t) \, dx = V_0
\]

where \( V \) is the initial volume \( V_0 = \ell h_0 \).
Generalization to higher dimensions: terminology

Terminology

We study evolution equations in the form:

$$U_t + A(U)U_x + B = 0,$$

with $A$ an $n \times n$ matrix. $B$ is a vector of dimension $n$ called the source. The system is homogeneous if $B = 0$. It is a conservative form when

$$U_t + \frac{\partial}{\partial x} F(U) = 0,$$

with $A(U) = \partial F/\partial U$.

The eigenvalues $\lambda_i$ of $A$ represent the speed(s) at which information propagates. They are the zeros of the polynomial $\det(A - \lambda 1) = 0$. The system is hyperbolic if $A$ has $n$ real eigenvalues.
If a function satisfies an evolution equation:

\[ u_t + [f(u)]_x = 0, \]

then we can create an infinity of equivalent PDEs:

\[ g(u)_t + [h(u)]_x = 0 \]

provided that \( g \) and \( h \) are such that \( h' = g'f' \). As long as the function \( u(x, t) \) is continuously differentiable, there is no problem, but for weak solutions (exhibiting a discontinuity), then the equations are no longer equivalent. We must use the original physical equation (usually expressing conservation of mass, momentum or energy).
Left and right eigenvectors

Take the particular case $n = 2$ for illustration. The matrix $A$ has two real eigenvalues $\lambda_1$ and $\lambda_2$ together with left eigenvectors $v_1$ and $v_2$:

$$v_i \cdot A = \lambda_i v_i.$$ 

It also has two right eigenvectors $w_1$ et $w_2$:

$$A \cdot w_i = \lambda_i w_i.$$ 

Let us assume that $A$ has the following entries

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
Then we get

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ d - a + \sqrt{\Delta} \\ 2c \end{pmatrix}, \quad w_1 = \begin{pmatrix} a - d + \sqrt{\Delta} \\ 2c \\ 1 \end{pmatrix}, \text{ associated with } \lambda_1 = \frac{a + d + \sqrt{\Delta}}{2}, \\
v_2 &= \begin{pmatrix} 1 \\ d - a - \sqrt{\Delta} \\ 2c \end{pmatrix}, \quad w_2 = \begin{pmatrix} a - d - \sqrt{\Delta} \\ 2c \\ 1 \end{pmatrix}, \text{ associated with } \lambda_2 = \frac{a + d - \sqrt{\Delta}}{2},
\end{align*}
\]

avec \( \Delta = (a - d)^2 + 4bc \). Note that

\[
v_1 \cdot w_2 = 0, \text{ and } v_2 \cdot w_1 = 0.
\]
Diagonalization

**Linear system:** When the eigenvectors are constant

\[ v_i \cdot U_t + v_i \cdot A(U)U_x + v_i \cdot B = 0. \]

thus:

\[ v_i \cdot U_t + \lambda_i v_i \cdot U_x + v_i \cdot B = 0. \]

We pose \( r_i = v_i \cdot U \) and obtain

\[ r_t + \Lambda \cdot r_x + r \cdot B = 0 \]

where \( \Lambda = \text{diag}\{\lambda_1, \lambda_2\} \). The system is now made of independent PDEs

\[ \frac{dr_1}{dt} + r_1 \cdot B = 0 \text{ along } x = x_{c,1}(t), \quad \frac{dx_{c,1}(t)}{dt} = \lambda_1, \]

\[ \frac{dr_2}{dt} + r_2 \cdot B = 0 \text{ along } x = x_{c,2}(t), \quad \frac{dx_{c,2}(t)}{dt} = \lambda_2, \]
Nonlinear system: We seek new variables \( \mathbf{r} = \{r_1, r_2\} \) such that:

\[
\mathbf{v}_1 \cdot d\mathbf{U} = \mu_1 d\mathbf{r}_1,
\]

\[
\mathbf{v}_2 \cdot d\mathbf{U} = \mu_2 d\mathbf{r}_2,
\]

where \( \mu_i \) are integrating factors such that \( d\mathbf{r}_i \) are exact differential. We have:

\[
\mu_1 d\mathbf{r}_1 = \mu_1 \left( \frac{\partial r_1}{\partial U_1} dU_1 + \frac{\partial r_1}{\partial U_2} dU_2 \right) = v_{11} dU_1 + v_{12} dU_2.
\]

Identifying the various terms leads to:

\[
\frac{\partial r_1}{\partial U_1} = \frac{v_{11}}{\mu_1},
\]

and

\[
\frac{\partial r_1}{\partial U_2} = \frac{v_{12}}{\mu_1}.
\]
Diagonalization: nonlinear system

By taking the ratio of the two equations above, we get:
\[
\frac{\partial r_1}{\partial U_1} = \frac{v_{11} \partial r_1}{v_{12} \partial U_2},
\]

The Schwartz theorem states that \( \partial_{xy} f = \partial_{yx} f \) and so from \( du(x, y) = adx + bdy \), we deduce that \( \partial_y a = \partial_x b \). Here this gives us the relation
\[
\frac{\partial}{\partial U_1} \frac{v_{12}}{\mu_1} = \frac{\partial}{\partial U_2} \frac{v_{11}}{\mu_1}.
\]

The integrating factor can also be deduced from \( \partial r_1 / \partial U_2 = 1/\mu_1 \) when the entries of \( v_1 \) are properly selected such that \( v_{11} = 1 \). Note that
\[
\frac{\partial r_1}{\partial U_1} = \frac{v_{11} \partial r_1}{v_{12} \partial U_2} \Rightarrow w_{21} \partial r_1 / \partial U_1 + w_{22} \partial r_1 / \partial U_2 = 0 \Rightarrow w_2 \cdot \nabla r_1 = 0
\]

**Definition:** \( r_1 \) is said to be a 2-invariant of the system.
The characteristic equation associated with the equation above is
\[
\frac{dU_1}{v_{12}} = \frac{dU_2}{v_{11}} = \frac{dr_1}{0},
\]
which leads to an integral. The first equation of the differential system is equivalent to:
\[
v_1 \cdot \frac{dU}{dt} \bigg|_{x=X_1(t)} + v_1 \cdot B = 0,
\]
where \( x = X_1(t) \) satisfies \( dX_1/dt = \lambda_1 \). This is the \( 1 \)-characteristic curve:
\[
\mu_1 \frac{dr_1}{dt} \bigg|_{x=X_1(t)} + v_1 \cdot B = 0.
\]
Similarly for \( r_2 \):
\[
\mu_2 \frac{dr_2}{dt} \bigg|_{x=X_2(t)} + v_2 \cdot B = 0.
\]
In a matrix form:

\[
\frac{dr}{dt} |_{r=X(t)} + S(r, B) = 0,
\]

along two characteristic curves \( r = X(t) \) such that \( \frac{dX(t)}{dt} = (\lambda_1, \lambda_2) \); \( S \) is the source term whose entries are \( \mu_i S_i = v_i \cdot B \). The new variables \( r \) are called the Riemann variables. For \( B = 0 \), they are constant along the characteristic curves and thus they are called Riemann invariants.
Exercise 2

Consider the Saint-Venant equations:

\[
\begin{align*}
\partial_t h + \partial_x (uh) &= 0, \\
\partial_t u + u \partial_x u + \partial_x h &= 0,
\end{align*}
\]

Determine the Riemann invariants and plot the characteristic curve for the dam-break problem

- initial velocity \(-\infty < x < \infty\) \(u(x, 0) = 0\)
- initial depth \(x < 0\) \(h(x, 0) = h_0\)
  \(x > 0\) \(h(x, 0) = 0\)
Consider the following linear hyperbolic problem:

$$\frac{\partial U}{\partial t} + A \cdot \frac{\partial U}{\partial x} = 0,$$

where $A$ is an $n \times n$ matrix with $n$ distinct real eigenvalues. We thus have $A = R \cdot \Lambda \cdot R^{-1}$, with $R$ the matrix associated with the change of coordinates (the columns are the right eigenvectors of $A$) and $\Lambda$ a diagonal matrix whose entries are $\lambda_i$. Making use of the change of variables $W = R^{-1} \cdot U$ leads to

$$\frac{\partial W}{\partial t} + \Lambda \cdot \frac{\partial W}{\partial x} = 0.$$

This is a system of independent linear hyperbolic PDEs: $\partial_t w_i + \lambda_i \partial_x w_i = 0$, whose solution takes the form $w_i = \omega_i(x - \lambda_i t)$. 

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The inverse change of variables leads to $eU = R \cdot W$

$$U = \sum_{i=1}^{n} w_i(x, t)r_i,$$

where $r_i$ is a right eigenvector $A$ associate to $\lambda_i$ and $w_i$ is $i$th entry of $W$. The solution results from the superimposition of $n$ waves travelling at speed $\lambda_i$; these waves are independent, do not change form (this form is given by the initial condition $\omega_i(x, 0)r_i$). When all but one elementary waves are constant ($\partial_x \omega_i(x, 0) = 0$), then the resulting wave is called a $j$-simple wave

$$U = \omega_j(x - \lambda_j t)r_j + \sum_{i=1, i \neq j}^{n} w_i(x, t)r_i,$$

Information propagates along the $j$-characteristic curve (all others $w_i$ are constant).
The Riemann problem: linear systems

The Riemann problem takes the form

$$\frac{\partial U}{\partial t} + A \cdot \frac{\partial U}{\partial x} = 0,$$

with

$$U(x, 0) = U_0(x) = \begin{cases} U_\ell & \text{if } x < 0, \\ U_r & \text{if } x > 0. \end{cases}$$

We now expand $U_\ell$ et $U_r$ in the eigenvector basis $r_i$

$$U_\ell = \sum_{i=1}^{n} w_i^{(\ell)} r_i \text{ et } U_r = \sum_{i=1}^{n} w_i^{(r)} r_i,$$

with $w_\ell = w_i^{(\ell)}$ et $w_r = w_i^{(r)}$ vectors with constant entries.
The Riemann problem involves \( n \) scalar problems

\[
 w_i(x, 0) = \begin{cases} 
 w^{(\ell)} & \text{if } x < 0, \\
 w^{(r)} & \text{if } x > 0. 
\end{cases}
\]

The solution to these advection equations is

\[
 w_i(x, t) = \begin{cases} 
 w^{(\ell)}_i & \text{if } x - \lambda_i t < 0, \\
 w^{(r)}_i & \text{if } x - \lambda_i t > 0. 
\end{cases}
\]

We call \( I(x, t) \) the largest index \( i \) such that \( x - \lambda_i t > 0 \). The solution reads

\[
 U(x, t) = \sum_{i=1}^{I} w^{(r)}_i r_i + \sum_{i=I+1}^{n} w^{(\ell)}_i r_i.
\]
Consider the case \( n = 3 \). The solution in the \( x - t \) space breaks down into "wedges" where \( U \) is constant and separated by characteristic curves \( x = \lambda_i t \). At any point \( M \), we can determine the value taken by \( U \) by plotting the characteristic curves issuing from \( M \) toward the \( x \)-axis.
Consider the wave equation

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \]

with initial data

\[ u(x, 0) = u_0(x) \text{ and } u_t(x, 0) = u_1(x) \]

Solve the equation.
The Riemann problem: nonlinear systems

Consider the following linear hyperbolic problem:

\[
\frac{\partial U}{\partial t} + A \cdot \frac{\partial U}{\partial x} = 0,
\]

where \( A = \nabla F \). This equation is invariant to the stretching group \((x, t) \rightarrow (\lambda x, \lambda t)\). We seek solutions in the form \( U(x, t) = W(\xi, u_L, u_R) \), with \( \xi = x/t \)

\[
-\xi \frac{dW}{d\xi} + \nabla F \cdot \frac{dW}{d\xi} = 0
\]

• \( W'(\xi) = 0 \), this is the constant state;

• \( W'(\xi) \) is a right eigenvector of \( \nabla F \) associate to \( \xi \) for all values taken by \( \xi \). The curve \( W(\xi) \) is tangent to the right eigenvector \( w \).
Generalizing the concept seen for 1D hyperbolic equations, we define a rarefaction wave as a simple wave function of $\xi = x/t$.

$$u(\xi) = \begin{cases} 
    u_L & \text{si } x/t \leq \xi_1, \\
    W(\xi, u_L, u_R) & \text{si } \xi_1 \leq x/t \leq \xi_2, \\
    u_R & \text{si } x/t \geq \xi_2.
\end{cases}$$

where $u_R$ and $u_L$ must satisfy $\lambda_k(u_L) < \lambda_k(u_R)$.
The Riemann problem: nonlinear systems

From the original PDE

\[-\xi \frac{dW}{d\xi} + \nabla F \cdot \frac{dW}{d\xi} = 0\]

we deduce that \( W' \) is a right eigenvector and that

\[\xi = \lambda_k(W),\]

and on differentiating with respect to \( \xi \), we get

\[1 = \nabla u \lambda_k(W) \cdot W'(\xi),\]

Since \( W' \) is a right eigenvector, \( W'(\xi) = \alpha w_k \), thus \( \alpha = [\nabla u \lambda_k(W) \cdot w_k]^{-1} \). The function \( W \) is solution to the ODE

\[W'(\xi) = \frac{w_k}{\nabla u \lambda_k(W) \cdot w_k},\]
Consider the following linear hyperbolic problem:
\[
\frac{\partial U}{\partial t} + A \cdot \frac{\partial U}{\partial x} = 0,
\]
where \( A = \nabla F \). This equation is invariant to the stretching group \((x, t) \rightarrow (\lambda x, \lambda t)\). We seek solutions in the form \( U(x, t) = W(\xi, u_L, u_R) \), with
\[
\xi = \frac{x}{t}
\]
\[
-\xi \frac{dW}{d\xi} + \nabla F \cdot \frac{dW}{d\xi} = 0
\]

- \( W'(\xi) = 0 \), this is the constant state;
- \( W'(\xi) \) is a right eigenvector of \( \nabla F \) associate to \( \xi \) for all values taken by \( \xi \). The curve \( W(\xi) \) is tangent to the right eigenvector \( w \).
The Riemann problem: nonlinear systems

A shock wave is a non-material surface $x = s(t)$ across which the solution is discontinuous $\dot{x} = s$. The Rankine-Hugoniot relation must hold

$$\dot{s}(u_L - u_R) = f(u_L) - f(u_R),$$

to which we add the Lax entropy condition

$$\lambda_k(u_L) > s > \lambda_k(u_L),$$

(jump in the $k$th field: we speak of a $k$-shock wave)
The Riemann problem: nonlinear systems

Summary

The solution to the Riemann problem:

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0,
\]

subject to

\[
U(x, 0) = U_0(x) = \begin{cases} 
U_L & \text{si } x < 0, \\
U_R & \text{si } x > 0.
\end{cases}
\]

involves \(n + 1\) states separated by \(n\) waves related to each eigenvalue.
For linear systems, the eigenvalues define shock waves. For nonlinear systems, different types of waves are possible:

- **shock wave**: in this case, the Rankine-Hugoniot holds

  \[ s' [U]_{x=s(t)} = F(U(x_L)) - F(U(x_R)) \]

  along with the entropy condition

  \[ \lambda_i(U_L) > s'_i > \lambda_i(U_R) \]

- **contact discontinuity** (when an eigenvalue is constant or such that \( \nabla_u \lambda_k \cdot w_k = 0 \)):

  the Rankine-Hugoniot relation holds, with the condition

  \[ \lambda_i(U_L) = \lambda_i(U_R) \]

- **rarefaction wave**: the characteristics fan out \( \lambda_i(U_L) < \lambda_i(U_R) \), self-similar solutions.
The Riemann problem: nonlinear systems

Hugoniot locus

The solution to the Riemann problem:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0,$$

subject to

$$U(x, 0) = U_0(x) = \begin{cases} U_L & \text{if } x < 0, \\ U_R & \text{if } x > 0. \end{cases}$$

involves $n + 1$ states separated by $n$ waves related to each eigenvalue.
Example: The Saint-Venant equations

Let us consider the Saint-Venant equations:

\[
\begin{align*}
\partial_t h + \partial_x (uh) &= 0, \\
\partial_t hu + \partial_x hu^2 + gh \partial_x h &= 0.
\end{align*}
\]

We introduce the unknowns \( U = (h, hu) \), the flux function \( F = (hu, hu^2 + gh^2/2) \) and the matrix \( A \):

\[
A = \frac{\partial F}{\partial U} = \begin{pmatrix}
0 & 1 \\
gh - u^2 & 2u
\end{pmatrix}.
\]

The conservative form is:

\[
\frac{\partial u}{\partial t} + A \cdot \frac{\partial u}{\partial x} = 0.
\]
**Example: The Saint-Venant equations**

Eigenvalues and eigenvectors for the conservative formulation (with \( c = \sqrt{gh} \))

<table>
<thead>
<tr>
<th>( i = 1 )</th>
<th>( i = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_i )</td>
<td>( u - c )</td>
</tr>
<tr>
<td>( w_i )</td>
<td>( \left{ \frac{1}{u - c}, 1 \right} )</td>
</tr>
<tr>
<td>( w_i \cdot \nabla \lambda_i )</td>
<td>( \frac{2(c - u)}{3c} )</td>
</tr>
</tbody>
</table>
Example: The Saint-Venant equations

If we take \((h, u)\) as variables, then the system is put in a nonconservative, but some solutions are easier to work out. With \(U = (h, u), F = (hu, hu^2 + gh^2/2)\) and matrix \(A\):

\[
A = \frac{\partial F}{\partial U} = \begin{pmatrix} u & h \\ g & u \end{pmatrix},
\]

\[
\frac{\partial u}{\partial t} + A \cdot \frac{\partial u}{\partial x} = 0.
\]
**Example: The Saint-Venant equations**

Eigenvalues and eigenvectors for the nonconservative formulation (with $c = \sqrt{gh}$)

<table>
<thead>
<tr>
<th></th>
<th>$i = 1$</th>
<th>$i = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>eigenvalues</strong></td>
<td>$\lambda_i$</td>
<td>$u - c$</td>
</tr>
<tr>
<td><strong>right eigenvectors</strong></td>
<td>$w_i$</td>
<td>$\left{ \frac{c}{g}, 1 \right}$</td>
</tr>
<tr>
<td><strong>left eigenvectors</strong></td>
<td>$v_i$</td>
<td>$\left{ -\frac{c}{h}, 1 \right}$</td>
</tr>
</tbody>
</table>

**Riemann invariants** $r_i$ | $u + 2c$ | $u - 2c$ |
Example: The Saint-Venant equations

Shock conditions

\[ \sigma[h] = [hu], \]
\[ \sigma[hu] = [hu^2 + gh^2/2], \]

with \( \sigma \) the shock velocity. In a frame related to the shock wave, then \( v = u - \sigma \) and

\[ h_1v_1 = h_2v_2, \]
\[ h_1v_1^2 + gh_1^2/2 = h_2v_2^2 + gh_2^2/2. \]

There are two families

- **1-shock**: \( \sigma < u_L - c_L \) et \( u_R - c_R < \sigma < u_R + c_R \). \( v_L > v_R \): the flux goes from left to right when \( v_L > 0 \);
- **2-shock**: \( \sigma > u_R + c_R \) et \( u_L - c_L < \sigma < u_L + c_L \). \( v_R > v_L \): the flux goes from right to left when \( v_L > 0 \).
Example: The Saint-Venant equations

Let us determine the Hugoniot locus, i.e., the points \((h_2, v_2)\) connected to \((h_1, v_1)\) by a 1- or 2-shock wave

\[
\sigma = \frac{h_2v_2 - h_1v_1}{h_2 - h_1},
\]

\[
\frac{(h_2u_2 - h_1u_1)^2}{h_2 - h_1} = h_2u_2^2 + \frac{gh_2^2}{2} - h_1u_1^2 - \frac{gh_1^2}{2},
\]

This gives us the shock speed and \(u_2(h_2|h_1\ v_1)\):

\[
u_2 = u_1 \mp (h_2 - h_1)\sqrt{\frac{gh_1 + h_2}{2h_1h_2}},
\]

\[
\sigma = u_1 \mp \sqrt{\frac{g}{2}}(h_1 + h_2)\frac{h_2}{h_1}.
\]
Example: The Saint-Venant equations

Rarefaction waves
We seek Riemann invariants $r_k$, defined as $\nabla_u r_k \cdot w_k = 0$. We work with the variables $(h, u)$. The first invariant is:

$$ -c \frac{\partial r}{\partial h} + \lambda_1 \frac{\partial r}{\partial u} = 0, $$

whose characteristic equations are

$$ \frac{du}{g} = -\frac{dh}{c}. $$

An integral is $u + 2c$. For the second invariant, we find $u - 2c$.

Along a 1-rarefaction wave, we have: $u_2 + 2\sqrt{gh_2} = u_1 + 2\sqrt{gh_1}$ and the invariant $r_1 = u + 2c$ is constant along any characteristic curve associated with the eigenvalue $\lambda_1 = u - c$ (when these fan out, $r_1$ is in the cone formed by the characteristics).
Example: The Saint-Venant equations

Show and rarefaction waves in the \((h, u)\) space. Arbitrarily the curves are issuing from \((h, u) = (1, 0)\)

Returning to the variables \((h, q = hu)\), we deduce

- Along a 1-rarefaction wave, we get:
  \[
  \frac{q_2}{h_2} + 2\sqrt{gh_2} = \frac{q_1}{h_1} + 2\sqrt{gh_1};
  \]

- Along a 2-rarefaction wave, we get:
  \[
  \frac{q_2}{h_2} - 2\sqrt{gh_2} = \frac{q_1}{h_1} - 2\sqrt{gh_1}.
  \]
Example: The Saint-Venant equations

Working out the solution to the Riemann problem

The construction method consists of introducing an intermediate state \( u_* \). The state \((h_*, u_*)\) can be connected to a left state \((h_L, u_L)\) through a 1-wave

\[
\begin{align*}
S_1(h_* \mid h_L, u_L) &= u_L + 2\sqrt{gh_L} - 2\sqrt{gh_*} & \text{if } h_* < h_L \quad \text{1-rarefaction wave} \\
R_1(h_* \mid h_L, u_L) &= u_L - (h_* - h_L)\sqrt{\frac{h_* + h_L}{2h_*h_L}} & \text{if } h_* > h_L \quad \text{1-shock wave}
\end{align*}
\]

It can be connected to a right state \((h_R, u_R)\) through a 2-wave

\[
\begin{align*}
S_2(h_* \mid h_R, u_R) &= u_R - 2\sqrt{gh_R} + 2\sqrt{gh_*} & \text{if } h_* < h_R \quad \text{2-rarefaction wave} \\
R_2(h_* \mid h_R, u_R) &= u_R + (h_* - h_R)\sqrt{\frac{h_* + h_R}{2h_*h_R}} & \text{if } h_* > h_R \quad \text{2-shock wave}
\end{align*}
\]
Example: The Saint-Venant equations

We begin with 1-waves, then 2-waves as information on the left gauche is primarily conveyed by the smallest eigenvalue, then the others.

Note that tangents to the curves $R_1$ et $S_1$ are the same. Note also that an intermediate state is possible only if:

$$u_R - u_L < 2(\sqrt{gh_R} + \sqrt{gh_L}).$$

For $h_L = 0$ ($h_R = 0$, resp.), then the 1-shock wave (the 2-shock wave, resp.) is undefined.

Solution to the Riemann problem for $(h_L, u_L) = (1, 0)$ et $(h_R, u_R) = (2, 0)$
Let us consider the (dimensionless) governing equations for a visco-elastoplastic material in a simple shear
\[
\frac{\partial u}{\partial t} = 1 + \frac{\partial \tau}{\partial z},
\]
\[
\frac{\partial \tau}{\partial t} = \frac{\partial u}{\partial z} - F(\tau),
\]
with \( F(\tau) = \max(0, |\tau| - 1)^{1/n} \tau / |\tau| \). The boundary and initial conditions are \( u = 0 \) at \( z = 0 \), \( \tau = 0 \) at \( z = 1 \), and \( \tau = u = 0 \) at \( t = 0 \). Cast the system into its characteristic form. Write a numerical code to solve the resulting system.