Similarity and Transport Phenomena in Fluid Dynamics
Homework

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Assistant: None

Deadline: 25 January 2019

Conditions: homework to be done alone. Please reply concisely to questions. There is no need to provide detailed explanations unless specifically requested. All software is authorized to answer questions. Students are encouraged to give back a typed copy.

Setting. The objective of this project is to study a paper by Lockington and coworkers on the Boussinesq equation (click on triangle to download the paper)

▼ Lockington, D.A., J.-Y. Parlange, M.B. Parlange, and J. Selker, Similarity solution of the Boussinesq equation, *Advances in Water Resources*, **23** (7), 725–729, 2000.

The Boussinesq equation is used to calculate the level of an aquifer in the ground; for example, as shown in Figure 1, a water flow in a channel, where the water height varies with time, causes groundwater flow. For one-dimensional problems, the Boussinesq equation reads

$$\theta \frac{\partial h}{\partial t} = K_s \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right), \tag{1}$$

with θ soil porosity, K_s (saturated) hydraulic conductivity, h height of the water table above a horizontal impermeable substrate. We consider the following boundary conditions

$$h(0,\ t) = h_0(t) = \sigma t^\alpha, \tag{2}$$

$$\lim_{x\to\infty} h(x,\ t) = 0, \tag{3}$$

with σ and α two constant parameters. The initial condition is

$$h(x,\ 0) = 0 \text{ for } x \geq 0. \tag{4}$$

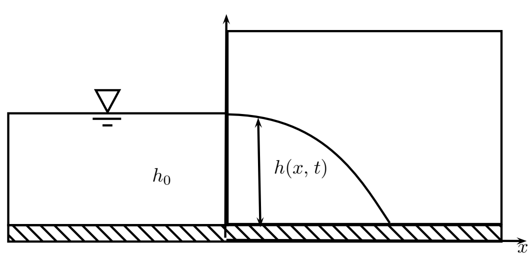

Figure 1: Spread of water into a porous medium.

Problem 1 Seeking similarity solutions

(a) Show that the Boussinesq equation (1) can be cast in the following dimensionless form

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right),$$

where h, x, t are now scaled variables.

Hint: introduce length, time, and height scales and find an appropriate timescale that simplifies the equation. Select the height scale such that the boundary condition becomes $H(0) = 1$ (we seek a similarity solution in the form $h = t^\gamma H(\xi)$ in the following).

(b) Show that the initial boundary-value problem (1–4) admits similarity solutions in the form $h = t^\gamma H(\xi)$ with $\xi = x/t^\beta$.

Hint: show that the equations are invariant to the stretching group. Deduce that similarity solutions can be worked out. Specify the values of β and γ as a function of α. Find the ordinary differential equation satisfied by H.

(c) Show that H satisfies

$$\frac{\mathrm{d}^2 H^2}{\mathrm{d}\xi^2} + (\alpha + 1)\xi\frac{\mathrm{d}H}{\mathrm{d}\xi} - 2\alpha H = 0. \tag{5}$$

Specify the initial conditions to which this equation is subject. Show that this equation is similar to Eq. (7) in Lockington's paper.

Hint: to show similarity with Lockington's equation, make use of the following change of variable: $\zeta = \sqrt{2(\alpha+1)}\xi$.

(d) How could you solve this equation numerically?

Hint: read through the paper by Lockington et al. and find an appropriate method.

Problem 2 Solving the reduced equation

(a) Show that Equation (5) is invariant to the one-parameter group of transformations

$$\xi \to \lambda\xi' \tag{6}$$

$$H \to \lambda^a H' \tag{7}$$

$$\frac{\mathrm{d}H}{\mathrm{d}\xi} \to \lambda^{a-1}\frac{\mathrm{d}H'}{\mathrm{d}\xi'} \tag{8}$$

$$\frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} \to \lambda^{a-2}\frac{\mathrm{d}^2 H'}{\mathrm{d}\xi'^2} \tag{9}$$

for any λ value.

Hint: find the value of a which makes Equation (5) invariant.

(b) Show that the governing differential equation in H can be reduced to the following quasi-linear first-order ordinary differential equation

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{2\alpha z - p(\alpha + 1 + 2p) - 2pz}{2z(p - 2z)}, \tag{10}$$

by making use of the following change of variable

$$z = \frac{H}{\xi^2} \text{ and } p = \frac{1}{\xi}\frac{\mathrm{d}H}{\mathrm{d}\xi}.$$

Hint: compute $\mathrm{d}p/\mathrm{d}\xi$, $\mathrm{d}z/\mathrm{d}\xi$, *and after substitution of* $\mathrm{d}^2H/\mathrm{d}\xi^2$, *derive the first-order ODE* $\mathrm{d}p/\mathrm{d}z$.

(c) Plot the phase portrait of Equation (10), with emphasis given to the fourth quadrant ($z > 0$ and $p < 0$). Show that there is a saddle point that represents the front position (i.e. a point x_f where $h(x_f) = 0$ and $h = 0$ beyond this point for the physical solution). Deduce the value p_f of p at that point.

Hint: write this equation in the form $\mathrm{d}p/\mathrm{d}z = Nu(p, z)/De(p, z)$ *with* Nu *and* De *two functions to be determined. Find the singular points and deduce the behavior of the integral path in the fourth quadrant. You can draw some portraits for particular* α *values and show the existence of a saddle point. You can also use the mathematica notebook available on the LHE website.*

(d) With these new pieces of information, we are now able to integrate (5) numerically. Provide a numerical simulation for $\alpha = -1/3$ and $\alpha = 1$. Start from an arbitrary value $\xi_f = 1$ and integrate (5) backwards over $[0, \xi_f]$. Plot the solution.

Hint: since you know that the front position is $(0, p_f)$ *in the* $z - p$ *plane, you can use these values to integrate (5) numerically using built-in functions in Mathematica (NDSolve) or MatLab (ode45).*

(e) Does your numerical simulation satisfy the boundary condition at $x = 0$ (or $\xi = 0$). How you can easily obtain the right solution, i.e., the solution satisfying $H(0) = 1$?

Hint: using the scaling invariance of (5), you can find an image of your numerical solution that satisfies the desired boundary condition.

(f) Find the front position for different α values and compare with Lockington's results.

Problem 3 Working in the $z - p$ plane

(a) You now have the numerical solution by another method. Justify why the separatrix $\mathcal{S}$ emanating from P (0, $-(\alpha + 1)/2$) is the only integral path that satisfies your boundary-value problem.

(b) What is the direction of this separatrix at point P?

Hint: using the techniques outlined in the lecture notes (see § A.3.1), find the slope m of the tangent at point P.

(c) Integrate (10) numerically to find the equation of the separatrix $\mathcal{S}$. Plot the separatrix in the the $z - p$ plane.

Hint: you cannot start integrating (10) from P since this point is singular. However, since you know the tangent of $\mathcal{S}$ at P, you can take a point lying on the tangent in the close neighborhood of P, e.g., P' (ϵ, $-(\alpha + 1)/2 - m\epsilon$).

(d) Deduce the numerical solution to (5).

Hint: From $\mathrm{d}z/\mathrm{d}\xi$, deduce the relation $\mathrm{d}\ln\xi = \mathrm{d}z/(p(z) - 2z)$. Using the numerical integral $p(z)$ of (10), integrate the relation to obtain $\xi(z)$. The parametric plot leads to the desired result.

(e) Compare the height profile obtained from the double integration of (5) and the profile you obtained in Problem 2. Do you think that this method is advantageous?

(f) How does the solution behave in the limit of $\xi \to 0$?

Hint: seek an asymptotic solution to (10) when $p \to -\infty$ and $z \to \infty$ by using dominant-balance arguments. Integrate the relation to obtain a new asymptotic relation for H when $\xi \to 0$.

Problem 4 Finding numerical solution with Matlab (optional problem)

(a) Using the built-in function pdepe of Matlab, solve the initial boundary-value problem (1–4).

(b) Find how the front position ξ_f varies with α. Compare with analytical results.