5.2

\[ 4 \frac{\partial^2 y}{\partial x^2} + 9 \frac{\partial y}{\partial x} + \frac{\partial y}{\partial x} = 0 \]  \hspace{1cm} (1)

IC
\[ \begin{align*}
\quad & y (0) = c_s t \\
\quad & y (\infty) = 0
\end{align*} \]

(a) \underline{Invariance?}

- translation \( y \rightarrow y + 2 \)
- stretching \( x \rightarrow Ax \)
  \[ y \rightarrow x^\beta y \]
  \[ 4 \frac{\partial^2 y}{\partial x^2} + 9 \frac{\partial y}{\partial x} \lambda^{1+\frac{5}{3}} \left( \frac{\beta}{\beta-1} \right) x \frac{\partial y}{\partial x}^{5/3} = 0 \]

so \[ \beta - 2 = \lambda^{1+\frac{5}{3}} \left( \frac{\beta}{\beta-1} \right) \]
\[ \beta \left( 1-\frac{5}{3} \right) = 3 - \frac{5}{3} \]
\[ \left[ \beta = \frac{-2}{1} \right] \]

\[ \Rightarrow \] There are two groups leaving \( \Gamma_1 \)

- invariant
- translation \( \Gamma_1 = \partial y \)
- stretching \( \Gamma_2 = x \partial_x - 2y \partial_y \)

(b) \underline{Solution using} \( \Gamma_1 \)

invariants given by
\[ \frac{dx}{\partial} = \frac{dy}{\partial} = \frac{dz}{\partial} \]
Reminder if \( \Gamma = \partial_y \)

Then \( \xi = 0 \) \( \eta = 1 \)
and so \( \gamma = \partial_x + \gamma_y \partial_y - 1 \gamma (\xi \partial_x + \xi_y \partial_y) = 0 \)
the one extended group is \( \Gamma^{(1)} = \partial_y \)

**Invariants**

\[
\frac{dx}{\lambda} = \frac{dy}{1} = \frac{dz}{\gamma_y} \quad \Rightarrow \text{two invariants}
\]

\[
\begin{align*}
\varphi &= x \\
\psi &= z
\end{align*}
\]

We make the change of variables

\( p = x, \; q = \gamma_y \)

\((1) \iff \frac{dq}{dp} = -\frac{9}{4} \; p \; q^{5/3}\)

\(\iff \frac{p \; dp}{q^{5/3}} + \frac{q \; dq}{9} = 0 \)

\(\iff \frac{1}{2} p^2 - \frac{2}{3} q^{-2/3} = C \quad (C \text{ constant of integration}) \)

\(q = (\frac{3}{4} p^2 + a)^{-1/2} \quad (2) \)

\((2) \Rightarrow \gamma_y = -\frac{\lambda}{(\frac{3}{4} p^2 + a)^{3/2}} \)

\(\gamma_y = b - \frac{2 \; x}{a \sqrt{4a + 3x^2}} \)
IC: \( y(z_0) = 0 \)

\[-b + \frac{2}{\sqrt{3}a} = 0 \Rightarrow b = \frac{2}{\sqrt{3}a},\]

\[y = \frac{2}{a} \left( 1 + \frac{1}{3} - \frac{2x}{\sqrt{4a^3 + 3x^2}} \right)\]

NB \( \frac{2x}{a\sqrt{4a^3 + 3x^2}} = \frac{2}{3} - \frac{4}{3\sqrt{3}} \frac{a}{a^3} + O(\frac{1}{x^4}) \)

Plot

(c) Solution using \( \Gamma_2 \)

Invariants given by

\[\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{-3z}\]

Remainder

\[\Gamma = x \frac{d}{x} - 2y \frac{d}{y} - \frac{dz}{z}\]

Then \( \gamma = x \) \( 0 = -2y \) \( \gamma = -3\gamma \)

Invariants

\[\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{-3z}\]

Note the 5 power in \( \gamma \). This we expect that \( \gamma > 0 \) and \( y > 0 \) (see Mathematica notebook)

Note the 5 power in \( \gamma \). This we expect that \( \gamma > 0 \) and \( y > 0 \) (see Mathematica notebook)
We define the invariants
\[ q = x^2 y^2, \]
\[ r = x^3 y. \]

\[
\begin{align*}
\frac{dp}{dx} &= 2 x q + x^2 y = (2p + q) \frac{y}{x} \\
\frac{dq}{dx} &= 3 x^2 y + x^3 q \\
&= \frac{1}{x} (3 q - x^2 y^2 - \frac{3}{4} x y^2) \\
&= \frac{1}{x} \left( 3 q - \frac{9}{4} \right) \left( \frac{9}{3} \right) \\
&= \frac{1}{x} \left( 3 - \frac{9}{4} \right) q^{\frac{2}{3}}
\end{align*}
\]

We take the ratio:
\[
\frac{dp}{dq} = \frac{q}{2p + q} \tag{3}
\]

Asymptotic solution

we seek \( y = A x + b = A x^{-2} \)

4 \((2)\) \((3)\) \( A x^{-4} = -9 x \left( -2 A x^{-3} \right)^{5/3} \)

\[
2 \left( \frac{A}{2} \right)^2 = -9 \left( -2 A \right)^{5/3}
\]

\[
2 \frac{A^2}{4} = + \left( \frac{5}{3} \right)^2 \Rightarrow A = \frac{4}{3^{\frac{5}{3}}}
\]
A: asymptotic point (saddle)
O: origin (node)

The solution is the separatrices.
To integrate numerically (3), we need
As to start from the vicinity of
\[ q_A = -2p_A, \quad p_A = A = \frac{413}{153} \]
because \( A \) is a saddle, and there is a single possible path through it.
To find the tangent of the separatrices at \( A \), we use L'Hôpital's rule
\[
m = \lim_{p \to p_A} \frac{q}{q_A + m(p-p_A)}
\]
Hence
\[
m = \lim_{p \to p_A} \frac{\frac{d}{dp}(p_A)}{\frac{d}{dp}(p_A)}
\]
We solve
\[ m = \frac{3}{2} \frac{3}{4} \frac{5}{3} m q^2 \]
and we find
\[ m = 0 \quad \text{or} \quad m = -4 \]
The only possibility is \( m = -4 \).
We deduce that the tangent of the separatrix at \( A \) is
\[ q = q_A + m (p - p_A) \quad \text{with} \quad m = -4 \]
Numerically, we solve (3) from \( q_0 = p_A - \varepsilon \)
to \( p = 0 \). The initial condition is
\[ q (p_0) = q_A + m \varepsilon \]
See Analysis. We obtain a function (interpolating polynomial)
\[ q = F (p) \]
Now to return to the original variable
we have to solve numerically
\[ y = x^{-3} F (y x^2) \]
with \( y (0) = 1 \) as initial condition.
**Remark 1**

Where does Eq. (1) come from?

Some models of superfluid helium assume that heat transfer is described by the nonlinear Gorster-Mellink law

\[ q = -k \left( \frac{2T}{\delta^2} \right)^{1/3} \]

(When the temperature gradient is negative, then

\[ q = R \left( \frac{-2T}{\delta^2} \right)^{1/3} \].

This leads to the nonlinear diffusion equation

\[ \frac{\partial T}{\partial t} = \frac{2}{\delta^2} \left( \frac{2T}{\delta^2} \right)^{1/3} \]

(after coordinate scaling). This equation is invariant to stretching groups. When the boundary condition takes the form \( T = T_0 \) at \( s = 0 \), the similarity solutions are sought in the form

\[ T = y(\xi), \quad \xi = \frac{s}{t^{3/4}} \]

The principal equation is

\[ \frac{4}{3} \frac{1}{\delta^3} \left( \frac{d\delta}{d\xi} \right)^{1/3} + \frac{\delta}{d\xi} \left( \frac{d\delta}{d\xi} \right) = 0 \]

which is equivalent to (1) \( 4\xi + 3x^2\xi^{5/3} = 0 \)
Remark 2

How to solve \((8)\) numerically?

\[4y' + 3x^2 y^{5/3} = 0\]
\[y(0) = a > 0\]
\[y(\infty) = 0\]

This is a two-point boundary value problem. The method of exact shooting seen in course does not hold here because the solution is a special solution (separatrix). This is the only integral path issuing from \(A\) in the phase portrait (asymptotic behaviour). Here we have to proceed by trial and error by assuming the value of \(y(0)\), set \(y(0) = a\), and solve \((8)\) numerically.