HANDBOOK OF DIFFERENTIAL EQUATIONS

Third Edition

Daniel Zwillinger
Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, NY

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The solutions of equations (122.2) and (122.3) are found to be (for $T/2 < t \leq T$)

$$u(t) = (\sinh\tau\sin\tau + \cosh\tau\cos\tau)\sin t + (\sinh\tau\cos\tau + \cosh\tau\sin\tau)\cos t,$$

and

$$v(t) = (\cosh\tau\sin\tau + \sinh\tau\cos\tau)\sin t + (\cosh\tau\cos\tau - \sinh\tau\sin\tau)\cos t,$$

where $\tau = T/2$. From these equations, we determine Δ to be

$$\Delta = u(T) + v'(T) = 2\cosh\tau\cos\tau. \tag{122.7}$$

The conclusion is that the solutions to equation (122.5) will be stable or unstable depending on whether the magnitude of Δ, as given by equation (122.7), is greater than or smaller than 2. Different values of T will give different conclusions. For example,

- If T=17 or $T = e^2$, then $|\Delta| > 2$ and some unstable solutions to equation (122.5) exist.
- If T=1 or $T = \pi$, then $|\Delta| < 2$ and all to the solutions to equation (122.5) are stable.

Notes

1. Mathematicians call this technique Floquet theory, whereas physicists call it Bloch wave theory. Solid state physicists use this technique to determine band gap energies.
2. Note that the periodicity of $f(t)$ in equation (122.5) does *not*, by itself, ensure that $y(t)$ has a periodic solution. If, however, $f(t)$ is periodic and has mean zero, then equation (122.5) will have a periodic solution of the same period.
3. The linear system $\mathbf{y}' = B(t)\mathbf{y}$ is said to be *noncritical* with respect to T if it has no periodic solution of period T except the trivial solution $\mathbf{y} = \mathbf{0}$. Otherwise, the system is said to be critical.
4. See also Coddington and Levinson [1, pages 78–81], Kaplan [3, pages 472–490], Lukes [5, Chapter 8, pages 162–179], and Magnus and Winkler [6, pages 3–10].

References

[1] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations.* McGraw–Hill Book Company, New York, 1955.

[2] Hassan, H. S. Floquet solutions of nonlinear ordinary differential equations. *Proc. Roy. Soc. Edin. Sect. A 106*, 3–4 (1987), 267–275.

[3] Kaplan, W. *Operational Methods for Linear Systems.* Addison–Wesley Publishing Co., Reading, MA, 1962.

[4] Kuchment, P. *Floquet Theory for Partial Differential Equations*, vol. 60 of *Operator Theory Advances and Appliations.* Birkhäuser, Basel, Switzerland, 1993.

[5] Lukes, D. L. *Differential Equations: Classical to Controlled.* Academic Press, New York, 1982.

[6] Magnus, W., and Winkler, S. *Hill's Equation.* Dover Publications, Inc., New York, 1966.

[7] Sleeman, B. D., and Smith, P. D. Double periodic Floquet theory for a second order system of ordinary differential equations. *Quart. J. Math. Oxford Ser. 37*, 147 (1986), 347–356.

123. Graphical Analysis: The Phase Plane

Applicable to Two coupled autonomous first order ordinary differential equations or an autonomous second order ordinary differential equation.

Yields

A graphical representation of the solution.

Idea

The qualitative features of the solution of two coupled autonomous first order ordinary differential equations may be ascertained from the phase plane.

Procedure

Suppose we have the set of two coupled autonomous first order ordinary differential equations

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y). \tag{123.1}$$

As t increases, $x(t)$ and $y(t)$ will describe a path in (x, y) space. This will not be the case at those points (x_0, y_0), where

$$f(x_0, y_0) = 0, \quad g(x_0, y_0) = 0.$$

At these points, the value does not change with t: $x(t) = x_0$ and $y(t) = y_0$. These points are called *critical points.* (They are also called *equilibrium points* or *singular points*).

To analyze the motion near a single critical point, we linearize equation (123.1) about that point. By a linear change of variables, we can place the critical point at the origin $(x, y) = (0, 0)$. Near a critical point at the origin, equation (123.1) can be written as

$$\begin{aligned} \frac{dx}{dt} &= ax + by + \widehat{f}(x, y), \\ \frac{dy}{dt} &= cx + dy + \widehat{g}(x, y), \end{aligned} \tag{123.2}$$

where $\widehat{f}(x, y) = o(|x| + |y|)$ and $\widehat{g}(x, y) = o(|x| + |y|)$ as $x \to 0$, $y \to 0$. We assume that a, b, c, d are real numbers and they are not all equal to zero. If we discard the $\widehat{f}$ and $\widehat{g}$ terms in equation (123.2) and look for solutions of the form

$$x(t) = Ae^{\lambda t}, \quad y(t) = Be^{\lambda t},$$

then we find that λ must be an eigenvalue of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. That is, λ must satisfy

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0. \tag{123.3}$$

There are five different types of behavior that can be observed near the critical point $(0, 0)$, based on the roots of equation (123.3). If the roots of equation (123.3) are

- Real, distinct, and of the same sign, then the critical point is called a *node*. (See figure 123.1.a for a typical picture.) Note that the symmetry axes are determined by the eigenvectors of the 2×2 matrix shown above.
- Real, distinct, and of opposite signs, then the critical point is called a *saddle point*. (See figure 123.1.b for a typical picture.)
- Real and equal, then the critical point is again a node. (See figure 123.1.c for a typical picture.)
- Pure imaginary, then the critical point is called a *center*. (See figure 123.1.d for a typical picture.)
- Conjugate complex numbers but not pure imaginary, then the critical point is called a *spiral* or a *focus*. (See figure 123.1.e for a typical picture.)

In each of the figures, an arrow points in the direction of increasing t. For each case illustrated, there exist systems in which the arrows are pointing in the opposite direction from what we have illustrated. Each solution of equation (123.2) (corresponding to different initial conditions) describes a single trajectory. Every trajectory must

- Go to infinity or
- Approach a limit cycle (see page 72) or
- Tend to a critical point

If the solution goes to infinity, then the solution is said to be *unstable* otherwise it is said to be *stable*.

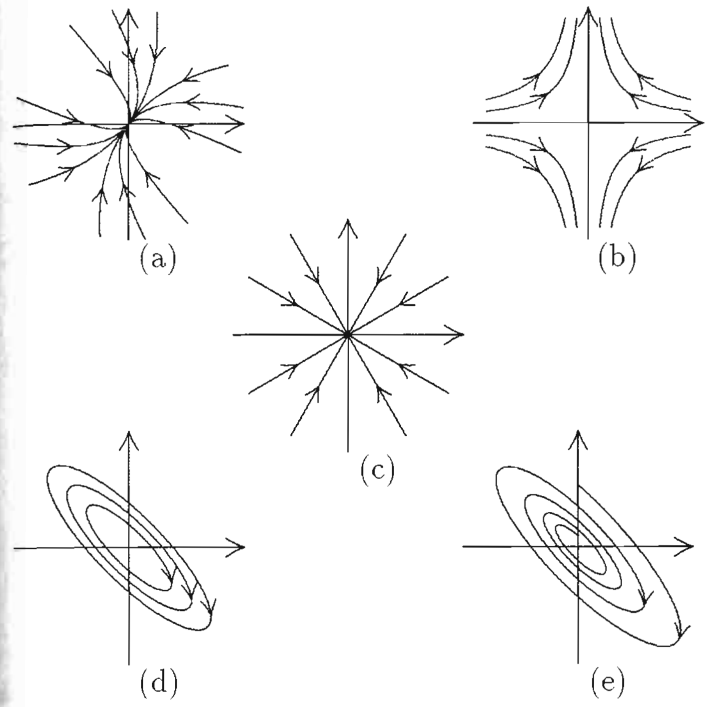

Figure 123.1: The different types of behavior in the phase plane: (a) and (c) are nodes, (b) is a saddle point, (d) is a center, and (e) is a spiral.

Example 1

Consider the simple linear differential equation system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}.$$

For this equation, the eigenvalues satisfy equation (123.3), which we write in the form $\lambda^2 - T\lambda + \Delta = 0$, where T is the trace of the matrix ($T = a + d$) and Δ is the determinant ($\Delta = ad - bc$). The eigenvalues, and the qualitative picture of the phase plane, can be deduced from T and Δ. Figure 123.2 shows the type of behavior to expect for different values of T and Δ. The curve figure 123.2 is given by determinant= (trace)^2; only centers can occur along this curve.

Example 2

Consider the nonlinear autonomous second order ordinary differential equation

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \omega^2 \sin x = 0, \tag{123.4}$$

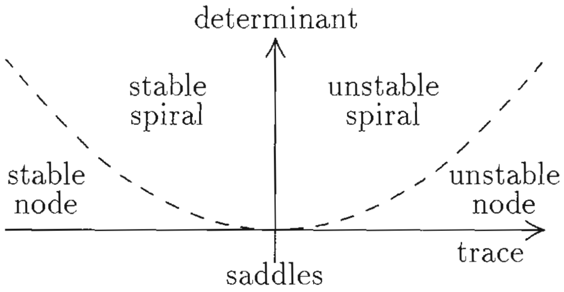

Figure 123.2: The different types of behavior in the phase plane, as a function of the trace and determinant of the 2×2 matrix.

which can be written as the coupled system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\beta y - \omega^2 \sin x. \end{aligned} \tag{123.5}$$

For the equations in equation (123.5) there are infinitely many critical points at the locations $\{x = n\pi,\ y = 0 \mid n = 0, 1, 2, \dots\}$. To analyze the behavior near the point $(k\pi, 0)$, the new variables $\widetilde{y} = y$, $\widetilde{x} = x - k\pi$ are introduced. In these new variables, the system in equation (123.5) can be approximated by

$$\begin{aligned} \frac{d\widetilde{x}}{dt} &= \widetilde{y}, \\ \frac{d\widetilde{y}}{dt} &= -\beta \widetilde{y} + (-1)^{k+1} \omega^2 \widetilde{x}, \end{aligned} \tag{123.6}$$

when $\widetilde{x}$ and $\widetilde{y}$ are both small. From equation (123.3) the characteristic equation for equation (123.6) becomes

$$\lambda^2 + \beta\lambda + \omega^2 (-1)^k = 0,$$

with the roots

$$\lambda_1 = \frac{-\beta + \sqrt{\beta^2 + (-1)^{k+1} 4\omega^2}}{2}, \quad \lambda_2 = \frac{-\beta - \sqrt{\beta^2 + (-1)^{k+1} 4\omega^2}}{2}.$$

If we now assume that $\beta > 0$ and $\beta^2 > 4\omega^2$, then

- For k even, $\lambda_1 < 0$ and $\lambda_2 < 0$. Hence, the point is a node.
- For k odd, $\lambda_1 > 0$ and $\lambda_2 < 0$. Hence, the point is a saddle point.

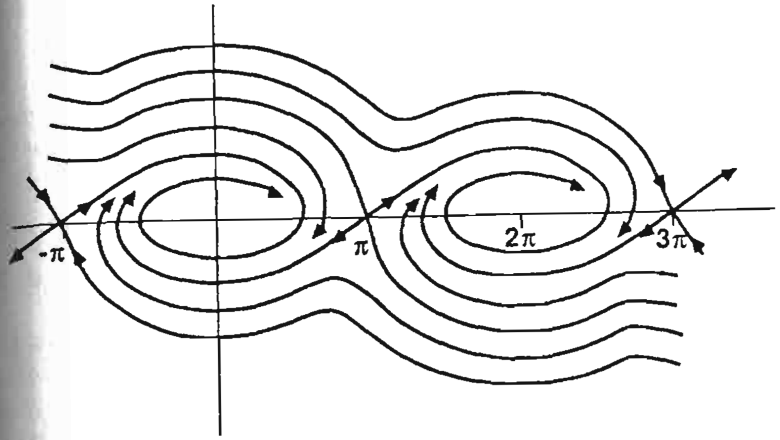

Figure 123.3: Phase plane for equation (123.4).

With this information, we can draw the phase plane for the system in equation (123.5) (see figure 123.3). Because the system in equation (123.4) is dissipative (i.e., the total "energy" decays), all of the different possible solutions approach one of the nodes in infinite time. The trajectories in the phase plane clearly show this.

Notes

1. In the above, we have presumed that the critical points are *isolated*; that is, each critical point has a neighborhood around it in which no other critical points are present.
2. If, in equation (123.2), $ad - bc$ were equal to zero, then second degree (or higher) terms in the Taylor series of f and g would be required to determine the behavior near that critical point. See Boyce and DiPrima [3, pages 456–486] for details.
   If $ad - bc \neq 0$, then the solution curves of the nonlinear system in equation (123.1) will be qualitatively similar to the solution curves of the linear system in equation (123.2), with the single exception that a center for equation (123.2) may be either a center or a spiral for the system in equation (123.1).
3. A second order autonomous ordinary differential equation can always be written as a first order system (see page 131). Also, the general equation of first order $M(x, y)\, dx + N(x, y)\, dy = 0$ may be written as a system in the form of equation (123.1); i.e.,

$$\frac{dx}{dt} = N(x, y), \qquad \frac{dy}{dt} = -M(x, y).$$

4. The point at infinity may be analyzed by changing variables by
$$x_1 = \frac{x}{x^2+y^2}; \qquad y_1 = \frac{-y}{x^2+y^2}$$
and then analyzing the point $(0,0)$ in the x_1, y_1-plane. This corresponds to the substitution $z_1 = 1/z$, when $z = x + iy$ is treated as a complex variable.
5. Kath [9] describes a method that combines phase plane techniques with matched asymptotic expansions. This method can be used to analyze second order, nonlinear, non-autonomous, singular boundary value problems.
6. Two different graphing programs for showing phase planes on a Macintosh computer are *DEGraph* and *Phase Portraits*. A review of these programs is in Hartz [5]. A program that runs on IBM personal computers (and compatibles) is *Phaser*; see Margolis [10] for a review.
7. A large collect of phase portraits may be found in Borrelli *et al.* [2].
8. See also Bender and Orszag [1, pages 171–197], Coddington and Levinson [4, Chapter 15, pages 371–388], and Huntley and Johnson [7, Chapter 8, pages 114–133].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill Book Company, New York, 1978.

[2] Borrelli, R. L., Coleman, C. S., and Boyce, W. E. *Differential Equations Laboratory Workbook.* John Wiley & Sons, New York, 1992.

[3] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[4] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations.* McGraw-Hill Book Company, New York, 1955.

[5] Hartz, D. Degraph and phase portraits. *Notices of the American Mathematical Society 36*, 5 (May/June 1989), 559–561.

[6] Hubbard, J., and West, B. *MacMath: A Dynamical Systems Software Package.* Springer-Verlag, New York, 1991.

[7] Huntley, I., and Johnson, R. M. *Linear and Nonlinear Differential Equations.* Halstead Press, New York, 1983.

[8] Jordan, D. W., and Smith, P. *Nonlinear Ordinary Differential Equations*, second ed. Clarendon Press, Oxford, England, 1987.

[9] Kath, W. L. Slowly varying phase planes and boundary-layer theory. *Stud. Appl. Math. 72* (1985), 2221–239.

[10] Margolis, M. S. Phaser. *Notices of the American Mathematical Society 37*, 4 (April 1990), 430–434.

[11] Wang, D. Computer algebraic methods for investigating plane differential systems of center and focus type. In *Computers and Mathematics*, E. Kaltofen and S. M. Watt, Eds. Springer-Verlag, New York, 1990, pp. 91–998.

124. Graphical Analysis: The Tangent Field

Applicable to First order ordinary differential equations.

Yields

A graphical representation of the solutions corresponding to different initial conditions.

Idea

The qualitative features of the solution of a first order ordinary differential equation may be ascertained from the tangent field.

Procedure

Given a first order ordinary differential equation in the form
$$\frac{dy}{dx} = f(x,y), \tag{124.1}$$
the procedure is to draw small line segments in the (x,y) plane, such that the line segment that goes through the point (x_0, y_0) has the slope $f(x_0, y_0)$. Note that a slope of m corresponds to an angle of $\tan^{-1} m$. After a region of (x,y) space has been covered with these small line segments, it should be apparent how the solution curves of equation (124.1) behave. An approximate solution may then be drawn by "connecting up" the line segments that originate from a given point.

Constructing the tangent field by hand is often facilitated by the *method of isoclines*. In this method, a few curves of the form $f(x,y) = C$, with C being a constant, are constructed. Along each one of these curves, dy/dx is equal to the constant C. Hence, at every point on these curves, the small line segments all have the same slope.

Example 1

Suppose we have the nonlinear ordinary differential equation
$$\frac{dy}{dx} = 1 - xy^2. \tag{124.2}$$
It is straightforward to construct the tangent field, which is shown in figure 124.1.

Every solution of equation (124.2) must be tangent to whatever line segments it passes near. For example, if equation (124.2) had the initial condition $y(0) = 1$, then the solution can be approximately traced by starting at the point $(0,1)$ and drawing a line that remains tangent to the line segments. For this equation and initial condition, y tends to zero as x tends to infinity. This behavior can be seen in figure 124.1.

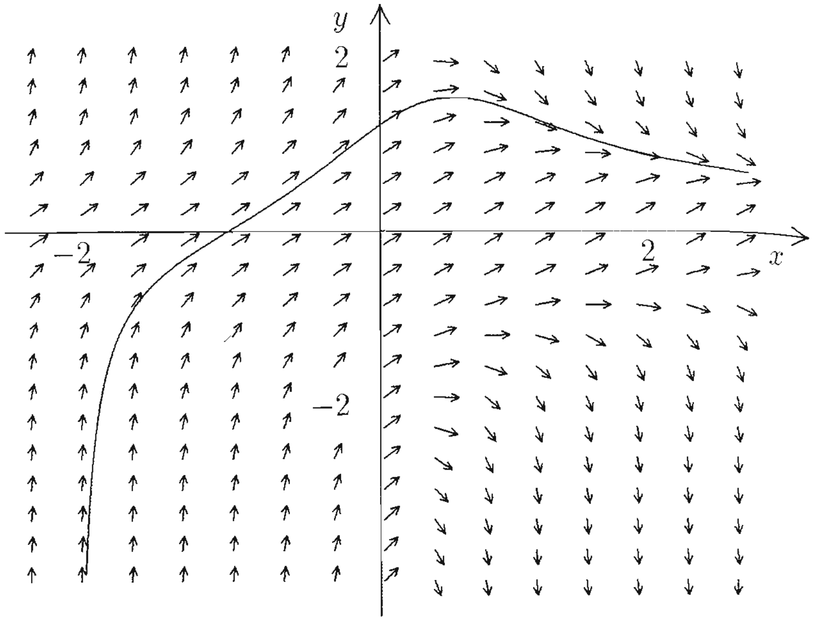

Figure 124.1: Tangent field for equation (124.2).

Example 2

Given the differential equation

$$\frac{dy}{dx} = 2x + y, \tag{124.3}$$

we find that the isoclines are the straight lines $2x + y = C$. Figure 124.2 shows the isoclines, with small line segments superposed, as well as three solutions to equation (124.3).

The exact solution to equation (124.3) is $y = 2(1 - x) + Ae^{-x}$, where A is an arbitrary constant. The linear behavior for $x \gg 0$ and the exponential behavior for $x < 0$ can be identified in this figure.

Notes

1. Consider drawing a small circle Γ in the (x, y) plane that surrounds the point (x_0, y_0). Traversing the circle counter-clockwise, the direction field will change. In every case, the change in angle must be a multiple of 2π: $[\text{angle}]_\Gamma = 2\pi I_\Gamma$, where I_Γ is an integer called the *index of the vector field*. Suppose the number of times the slope dy/dx changes from $+\infty$ to $-\infty$ is m and number of times it changes from $-\infty$ to $+\infty$ is n. Then the index is equal to $(m - n)/2$. The index may be positive, negative, or zero. If Γ surrounds no critical points, then the index is zero. If Γ surrounds a saddle point, then the

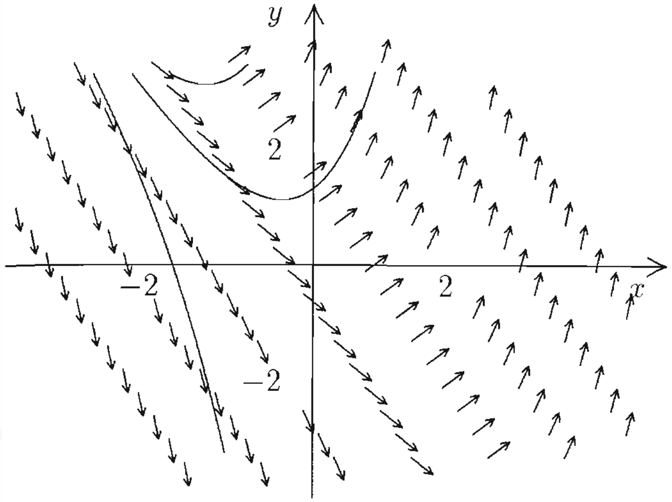

Figure 124.2: Tangent field for equation (124.3).

index is -1. If Γ surrounds a center, spiral, or node, then the index is $+1$. If Γ surrounds more than one critical point, then the index is the sum of the indices for each critical point.
Equation (124.1) sometimes arises from the autonomous system $\{\dot{x} = F(x, y),\ \dot{y} = G(x, y)\}$, via $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$. In this case, we have $I_\Gamma = \frac{1}{2\pi}\oint_\Gamma \frac{F\,dG - G\,dF}{F^2 + G^2}$. See Jordan and Smith [3] for details.
2. Mathematica has the packages `PlotField` and `PlotField3D` which can plot two- and three-dimensional vector fields. They contain functions for plotting gradient and Hamiltonian vector fields.
3. Even rough hand construction of the tangent field can produce useful qualitative information.
4. See also Bender and Orszag [1, pages 148–149] and Boyce and DiPrima [2, pages 34–35].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw–Hill Book Company, New York, 1978.
[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.
[3] Jordan, D. W., and Smith, P. *Nonlinear Ordinary Differential Equations*, second ed. Clarendon Press, Oxford, England, 1987.