Flow down inclined channel as a discriminating experiment Research report 1

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Flows down inclined channel may be seen as a rheometric test. We suggest using such a flow geometry as a discriminating experiment to test the reliability of various constitutive equations proposed for modeling a single-phase flow or to deduce the main rheological properties. To that purpose, we have endeavored to link the main bulk properties to the constitutive equation characteristics. Investigated properties include the velocity profile, the discharge equation, etc. Using an analogy with hydraulics, we examine the form of the free surface profile, the existence of sub- and supercritical flows, the development of bores (jumps), and the (linear) stability domain. We have applied the results to the case of Newtonian fluids. The obtained results can serve a guide to interpret experimental data for various types of complex fluids.

1. Introduction

Many experiments are performed in laboratory using an inclined channel since this geometry is very close to practical situations encountered in industry or environment. In hydraulics, a large amount of literature is devoted to the computation of flows down channels (Chow 1959). The development of instabilities (roll waves, transition from laminar to turbulent flows, etc.) has been also studied using channel as flow geometry. On rare occasions, a channel has been used as a rheometer. Examples include the investigations on the rheological properties of polymeric liquids (Astarita & Palumbo 1964), granular materials (Ancey *et al.* 1996; Suzuki & Tanaka 1971), and clay suspensions (Coussot 1995). More frequently, the reliability of some constitutive equations proposed in the literature is tested by comparing theoretical predictions and experimental data obtained on channels. For instance, several comparisons have been made on the velocity profile of flow down inclined channel to test the reliability of kinetic theories for granular materials (Ahn *et al.* 1991).

The underlying idea of the present report is to arrive at a quite complete theoretical description of the connection between local properties (among others the bulk constitutive equation) and flow features (velocity profiles, discharge equation, flow stability, etc.). On the basis of this connection, it is possible to consider flows down inclined planes (in practice, channels) as a discriminating experiment, which allows to test the reliability of the different constitutive equations proposed for modelling a single-phase flow. In the following, we shall examine the general features of uniform flows and non-uniform flows down inclined channels. In the former case, we shall determine the main flow properties such as the velocity and density profiles, and the discharge equation. In the latter case,

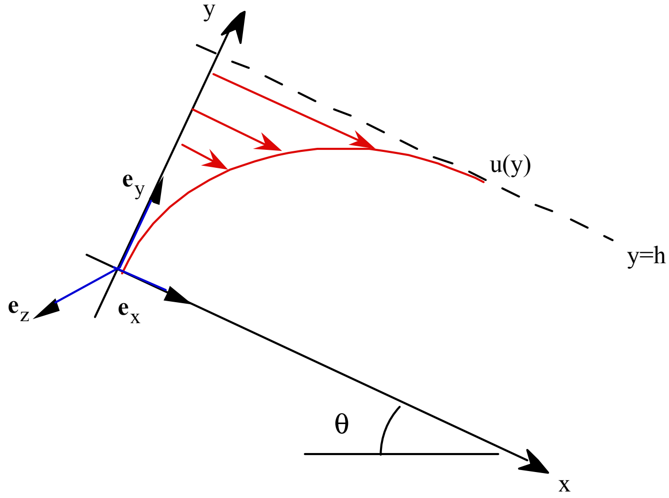

Figure 1. Definition sketch for steady uniform flow.

an analogy with hydraulics shall be drawn: emphasis will be given to free surface curve, existence of sub- and super-critical flows, formation of jump, flow stability. In both cases, we endeavor to analytically relate bulk flow and local properties.

The originality of the report is to propose a consistent and general framework to study the rheological properties of various materials. The results obtained in the report may be applied to a broad class of materials. We shall only consider situations in which the fluid flow can be treated as isochoric (at least as a first approximation). Some types of material (two-phase material, strongly compressible material, etc.) have to be discarded since their description is outside of the customary treatment of single-phase continua or are not compatible with assumption of isochoric flow. We will apply these results to the classical well-known case of Newtonian fluids.

2. Equations of motion and flow characteristics

2.1. *Governing equations for a flow down an inclined plane*

In this section, we focus attention on gravity-driven free-surface flows of a fluid down an inclined plane (rectilinear flow). It is assumed that (i) a steady uniform regime can take place at an inclination θ to the horizontal, (ii) the continuum undergoes a simple shear. We use the Cartesian co-ordinate system of origin 0 and of basis $\mathbf{e}_x$, $\mathbf{e}_y$, as depicted in Figure 1.

The kinematic field depends on the co-ordinate y alone and takes the following form

$$v_x = u(y), \quad v_y = 0, \quad v_z = 0, \tag{2.1}$$

where u is a function of y to be determined. Accordingly the strain-rate tensor $\bar{\mathbf{d}}$ has the following components in the co-ordinate system

$$\bar{\mathbf{d}} = \frac{\dot\gamma}{2}\left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \tag{2.2}$$

where $\dot\gamma$ denotes the shear rate; it is defined as a function of the co-ordinate y and implicitly of the inclination θ

$$\dot{\gamma}(y) = \left(\frac{\partial u}{\partial y}\right)_{\theta}. \tag{2.3}$$

The momentum balance can be written as

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho \mathbf{g} + \nabla \cdot \mathbf{\Sigma}, \tag{2.4}$$

where ρ and $\mathbf{g}$ respectively denote the local material density and the gravitational acceleration. As sign convention, we use positive stress to represent tensile stress. $\mathbf{\Sigma}$ stands for the total stress tensor. Without restriction, the stress tensor can be written as the sum of a spherical tensor and a deviatoric term called the extra-stress tensor ($\mathbf{T}$) (Coleman *et al.* 1966; Tanner 1988):

$$\mathbf{\Sigma} = -p\mathbf{1} + \mathbf{T}, \tag{2.5}$$

where $\mathbf{1}$ denotes the identity tensor and p is a scalar referred to as pressure. Two complementary classes of materials can be represented by the relation (2.5). The first class corresponds to compressible materials, for which the pressure is defined thermodynamically (using the free energy). The second class includes incompressible materials, for which the pressure is indeterminate and is found by solving the equations of motion. In this case, in order to remove the non-uniqueness of $\mathbf{T}$ (due to the indeterminate pressure), the following convention is usual: $\mathrm{tr}\,\mathbf{T} = 0$. For a homogeneous and isotropic simple fluid, the extra-stress tensor depends on the strain-rate only: $\mathbf{T} = G(\mathbf{D})$, where G is an isotropic functional. Many classes of material are in fact neither homogeneous, nor isotropic, nor merely expressible in the form given equation (2.5). In that case, the extra-stress tensor may be a function of the strain-rate tensor and other parameters which can be scalar or tensorial quantities (Ψ): $\mathbf{T} = G(\mathbf{D}, \Psi)$. Further equations are needed to supplement the governing equations. A typical example is given by kinetic theories where the extra-stress tensor is written as a function of the strain-rate and the temperature (Campbell 1990). The temperature is a scalar parameter, whose variations are governed by the (kinetic) energy balance equation. In some cases, the constitutive equation may be expressed in the form given by equation (2.5) only for a steady state. This is the case for materials with time-dependent properties (thixotropic materials, viscoelastic materials). In practice, we are not dealing with the stress tensor in its whole. It is often more convenient to use its components. We shall focus on the shear stress $\tau = \Sigma_{xy} = T_{xy}$ and the normal stress differences

$$N_1 = \Sigma_{xx} - \Sigma_{yy} \text{ and } N_2 = \Sigma_{yy} - \Sigma_{zz},$$

called the first and second normal stress differences, respectively.

2.2. *Boundary conditions*

We have to specify the boundary conditions for stress and velocity fields at the free surface and at the bottom wall. We assume that there is no slip at the bottom: $u(y) = 0$. Furthermore, we assume that there is no interaction between the free surface and the ambient fluid above (except the pressure exerted by the ambient fluid). Notably, we ignore surface tension effects on the free surface. If the boundary condition at the free surface suffers no criticism, the boundary condition at the bottom appears more delicate. For many non-Newtonian fluid, the no-slip condition is not satisfied (Barnes 1995). In practice, this imposes either to measure the slip velocity and to take it into account in the computations or to use suitable devices. For instance, the use of roughened base is generally sufficient to vanish the slip velocity, but in turn can entail new disturbing phenomena and the development of a boundary layer with rheological specificities different from the ones of the studied material. For instance, for a suspension of particles, this is reflected by depletion, torque transmission, energy flux, etc. It is expected that for sufficiently thick flows, such phenomena have minute effects on the whole dynamics. In the simplified approach below, they will be neglected.

2.3. *Properties of steady uniform flows*

Since for steady rectilinear flows, acceleration vanishes and the components of $\mathbf{T}$ depend on y alone, the equations of motion (2.4) reduce to

$$0 = \frac{\partial T_{xy}}{\partial y} - \frac{\partial p}{\partial x} + \rho g \sin\theta, \tag{2.6}$$

$$0 = \frac{\partial T_{yy}}{\partial y} - \frac{\partial p}{\partial y} - \rho g \cos\theta, \tag{2.7}$$

$$0 = \frac{\partial p}{\partial z}. \tag{2.8}$$

It follows from equation (2.8) that p is independent of z. Accordingly, integrating (2.7) between y and h implies

$$p(x,y) - p(x,h) = T_{yy}(y) - T_{yy}(h) + g\cos\theta \int_y^h \rho(y)\mathrm{d}y. \tag{2.9}$$

We can express equation (2.6) in the following form

$$\frac{\partial}{\partial y}\left(T_{xy} + g\sin\theta \int_y^h \rho(y)\mathrm{d}\mathrm{d}y\right) = \frac{\partial p(x,h)}{\partial x}. \tag{2.10}$$

The only solution is found by noting that both terms of this equation must equal to a function of z, which is denoted by $b(z)$. Moreover, equation (2.8) implies that b is actually independent of z; thus, in the following we shall note: $b(z) = b$. The solutions to equation (2.10) are

$$p\,(x,h) = bx + c, \tag{2.11}$$

$$T_{xy}(h) - T_{xy}(y) - g\sin\theta \int_y^h \rho(y)\mathrm{d}y = b(h-y), \tag{2.12}$$

where c is a constant that we shall determine. To that purpose, let us consider the free surface. It is reasonable and usual to assume that the ambient fluid friction is negligible. The stress continuity at the interface implies that the ambient fluid pressure p_0 exerted on an elementary surface at $y = h$ (oriented by $\mathbf{e}_y$) must equal the stress exerted by the fluid. Henceforth, the boundary conditions at the free surface may be expressed as

$$-p_0\mathbf{e}_y = \mathbf{\Sigma}\cdot\mathbf{e}_y, \tag{2.13}$$

which implies in turn that

$$T_{xy}(h) = 0, \tag{2.14}$$

$$p_0 = p(x,h) - T_{yy}(h), \tag{2.15}$$

Comparing equations (2.15) and (2.11) leads to $b = 0$ and $c = p_0 + T_{yy}(h)$. Accordingly, from equations (2.12) and (2.9), we obtain for the shear and normal stress distributions

$$\tau = T_{xy} = g\sin\theta \int_y^h \rho(y)\mathrm{d}y \approx \bar{\rho} g \sin\theta (h-y), \tag{2.16}$$

$$\Sigma_{yy} = T_{yy} - (p - p_0) = -g\cos\theta \int_y^h \rho \mathrm{d}y \approx -\bar{\rho} g \cos\theta (h - y), \tag{2.17}$$

in which we have neglected the variations in density. As a first approximation, the local density is replaced by its mean value: $\rho \approx \bar{\rho}$. The shear and normal stress profiles are determined regardless of the form of the constitutive equation. This property is of great interest and motivates the use of flow down channel as a rheometric test. In many cases (simple fluids), the shear stress is a one-to-one function of the shear rate: $\tau = f(\dot{\gamma})$. Using the shear stress distribution (2.16) and the inverse function f^{-1}, we find

$$\dot{\gamma} = f^{-1}(\tau). \tag{2.18}$$

A simple integration leads to the velocity profile

$$u(y) = \int_0^y f^{-1}(\tau(\xi))\mathrm{d}\xi. \tag{2.19}$$

A new integration gives the flow rate (per unit width)

$$q = \int_0^h \int_0^y f^{-1}(\tau(\xi))\mathrm{d}\xi. \tag{2.20}$$

The relationship between the flow rate and the flow depth is called the discharge equation. The measurement of the velocity profile and the calculation of its derivative can provide rheological information on the tested material: since the shear stress distribution is imposed and known, it is possible to relate this distribution and the shear rate profile, and thus to obtain an estimate of the relationship: $\tau = f(\dot{\gamma})$. An alternative and simpler method exists for getting information on the constitutive equation. When the discharge equation has been inferred from experimental data, the inverse procedure may be used to estimate the relationship between the shear rate and shear stress. Indeed, using an integration by part, it is possible to express the flow rate in the following way

$$q = h\, u_g + \int_0^h f^{-1}(\tau(\xi))(h - \xi)\mathrm{d}\xi, \tag{2.21}$$

where u_g denotes the slip velocity, which is assumed to be zero. The shear stress distribution is given by equation (2.16). Taking the partial derivative of q with respect to h, we obtain

$$\dot{\gamma} = f^{-1}(\tau(h)) = \frac{1}{h}\left(\frac{\partial q}{\partial h}\right)_\theta. \tag{2.22}$$

This relation allows us to directly use a channel as a rheometer. The other (normal) components of the stress tensor cannot be easily measured. The curvature of the free surface of a channelized flow may give some indications on the first normal stress difference. Let us imagine the case where it is not equal to zero. Substituting the normal component Σ_{yy} by Σ_{xx} into equation (2.7), we find after integration

$$T_{xx} = p + \rho g y \cos\theta + N_1 + c, \tag{2.23}$$

where c is a constant. Imagine that a flow section is isolated from the rest of the flow and the adjacent parts are removed. In order to hold the free surface flat (namely it will be given by the equation $y = h$, $\forall z$), the normal component Σ_{xx} must vary and balance the variations in N_1 due to the presence of the sidewalls (for a given depth, the shear rate is higher in the vicinity of the wall than in the center). But at the free surface, the boundary condition (2.13) compels the normal stress Σ_{xx} to vanish and the free surface to bulge out. To first order, the free surface equation is

$$-\rho g y \cos\theta = N_1 + c. \tag{2.24}$$

If the first normal stress difference vanishes, the boundary condition (2.13) is automatically satisfied and the free surface is flat. In the case where the first normal stress difference does not depend on the shear rate, there is no curvature of the shear free surface. The observation of the free surface may be seen as a practical test to examine the existence (and sign) of the first normal stress difference and to quantify it (by measuring both the velocity profile at the free surface and the free surface equation).

In most cases where complex fluids are involved, the shear stress is not a one-to-one function of the shear rate, since parameters of the constitutive equations Ψ may vary as functions of the shear rate. A similar treatment to the one above is still possible but, if it enlightens the rheological behavior of the tested material, it does not allow to fully deduce the constitutive equation. We replace the constitutive equation $\tau = f(\dot\gamma, \Psi(\dot\gamma))$ by the relaxed equation: $\tau = f(\dot\gamma, \Psi(\dot\gamma)) = g(\dot\gamma)$. The function is not intrinsic to the material and depends on the boundary conditions and channel geometry, but its main characteristics are still expected to reflect the rheological behavior in a steady state. On the basis of this approximation, it is possible to proceed as previously with $f(\dot\gamma)$.

2.4. *Properties of steady non-uniform flows*

The solution to the problem of steady uniform flow down channel can only provide limited information on the rheological behavior. In practice, experiments are not carried out down infinite inclined planes, but on the contrary down channels of finite size. The finite size of the channel lead to difficulties, which demand care when merely using a channel as a rheometer. However, in many cases, the deviation from the steady uniform flow solution originates from the rheological behavior and the idea developed here is to interpret it in terms of rheological properties. An example has been given above with the use of the free surface curvature to estimate the first normal stress difference. To go further in the investigation of the rheological properties, it is necessary to examine nonuniform flow properties. Here, we shall address the issue of slightly nonuniform flows. Attention is paid on gradually varied flows only, namely flows steady for the time interval under consideration and with approximately parallel streamlines. In this context, the equations of motion may be inferred in a way similar to the usual procedure used in hydraulics to derive the shallow water equations (or Saint-Venant equations): it consists of integrating the momentum and mass balance equations over the depth. As such a method has been extensively used in hydraulics for water flow (Chow 1959) as well for non-Newtonian fluids [see for instance Savage (1991) or Piau (1996)], we briefly remind the principle and then directly provide the resulting governing equations. In contrast with hydraulics, care must be paid here on the possible variations in the flow density. Let us consider the local mass balance in a steady state: $\nabla \cdot (\rho \mathbf{u}) = 0$. Integrating over the flow depth leads to

$$\int_0^{h(x,t)} \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}\right) \mathrm{d}y = \frac{\partial}{\partial x}\int_0^h \rho u(x,y,t)\mathrm{d}y - \rho(h)\left(u(h)\frac{\partial h}{\partial x} + v(x,h,t)\right) - \rho(0) v(x,0,t), \tag{2.25}$$

where u and v denote the x- and y-component of the local velocity. At the free surface and the bottom, the y-component of velocity satisfies the following boundary conditions

$$v(h) = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + u(x,h,t)\frac{\partial h}{\partial x}, \tag{2.26}$$

$$v(x, 0, t) = 0. \tag{2.27}$$

We easily deduce

$$\rho(h)\frac{\partial h}{\partial t} + \frac{\partial h\overline{\rho u}}{\partial x} = 0, \tag{2.28}$$

where we have introduce depth-averaged values defined as

$$\bar{f}(x,t) = \frac{1}{h(x,t)} \int_0^{h(x,t)} f(x,y,t)\mathrm{d}y. \tag{2.29}$$

The same procedure is applied to the momentum balance given in equation (2.4). Without difficulty, we can deduce the averaged momentum equation

$$\frac{\partial h\overline{\rho u}}{\partial t} + \frac{\partial h\overline{\rho u^2}}{\partial x} + h\overline{\rho u \nabla \cdot \mathbf{u}} = \bar{\rho} g h \sin\theta + \frac{\partial h\bar{\Sigma}_{xx}}{\partial x} - \tau_p \tag{2.30}$$

where we have introduced the bottom shear stress: $\tau_p = T_{xy}(x, 0, t)$. Let us notice that the third term in the left-hand side of equation (2.30) ($h\overline{\rho u \nabla . \mathbf{u}}$) does not exist in hydraulics due to water incompressibility. In the present form, the equation system (2.30) and (2.28) is not closed since the number of variables exceeds the number of equations. As previously, a major simplification is brought by assuming that dilatancy only causes slight variations in the bulk density and accordingly the bulk density can be replaced by its mean value. Another simplification consists in introducing a parameter α (sometimes called the Boussinesq momentum coefficient) which links the mean velocity to the mean square velocity

$$\overline{u^2} = \frac{1}{h}\int_0^h u^2(y)\,\mathrm{d}y = \alpha \bar{u}^2. \tag{2.31}$$

This leads to the following system of equations, put into a conservative form

$$\frac{\partial \mathbf{S}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{S})}{\partial x} = \mathbf{G}^*(\mathbf{S}) \Leftrightarrow \frac{\partial \mathbf{S}}{\partial t} + \nabla \mathbf{F} \cdot \frac{\partial \mathbf{S}}{\partial x} = \mathbf{G}^*(\mathbf{S}), \tag{2.32}$$

in which we have introduced

$$\mathbf{S} = \left[\begin{array}{c} h \\ \bar{u} \end{array}\right],\ \mathbf{G} = \nabla \mathbf{F} = \left[\begin{array}{cc} \bar{u} & h \\ (\alpha - 1)\frac{\bar{u}^2}{h} - \frac{1}{\bar{\rho} h}\frac{\partial h\bar{\Sigma}_{xx}}{\partial h} & \bar{u}(2\alpha - 1) - \frac{1}{\bar{\rho} h}\frac{\partial h\bar{\Sigma}_{xx}}{\partial \bar{u}} \end{array}\right],\ \mathbf{G}^* = \left[\begin{array}{c} 0 \\ g\sin\theta - \frac{\tau_p}{\bar{\rho} h} \end{array}\right].$$

Another helpful (and usual) approximation, not mentioned in the system above, concerns the computation of stress (Chow 1959). Putting ourselves in the framework of long wave approximation, we assume that longitudinal motion outweighs vertical motion: for any quantity m related to motion, we have $\partial m/\partial y \gg \partial m/\partial x$. This allows one to consider that every slice of flow can be treated as if it was locally uniform. A way of justifying this approximation is to consider the local equations of motion in a dimensionless form

$$\left\{\begin{array}{l} \varepsilon\dfrac{d\tilde{u}}{\mathrm{d}\tilde{t}} = \varepsilon\dfrac{\partial \tilde{\Sigma}_{xx}}{\partial \tilde{x}} + \dfrac{\sin\theta}{Fr^2} + \dfrac{\partial \tilde{\tau}}{\partial \tilde{y}} \\ \varepsilon\dfrac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{t}} = \varepsilon\dfrac{\partial \tilde{\tau}}{\partial \tilde{y}} + \dfrac{\partial \tilde{\Sigma}_{yy}}{\partial \tilde{x}} - \dfrac{\cos\theta}{Fr^2} \end{array}\right\}, \tag{2.33}$$

where ε is the ratio H/L, in which H and L are, respectively, the orders of magnitude of the flow depth and length. The x-component of velocity (u) has been scaled by a reference velocity U_0. The mass balance equation implies that the y-component of velocity (v) must be scaled by L. For stress, we have used the ratio $\Sigma/(\rho U_0^2)$. Space and time dimensionless variables are chosen as follows: $\tilde{x} = x/L$, $\tilde{y} = y/H$, $\tilde{t} = tU_0/L$. We have also introduced the Froude number: $Fr = U_0/\sqrt{gH}$. Assuming that each stress field may be expanded into a power series in ε (as follows: $\tilde{\tau} = \tilde{\tau}_0 + \varepsilon\tilde{\tau}_1 + \varepsilon^2\tilde{\tau}_2 + ...$, where $\tilde{\tau}_0$ corresponds to the shear stress for a steady uniform flow) then collecting the powers of ε leads to a sequence of differential equations for the functions $(\tilde{\tau}_i)$. For instance, for terms of order ε^1, we find

$$\frac{\partial \tilde{\tau}_1}{\partial \tilde{y}} = 0,$$

and

$$\frac{\partial \tilde{\Sigma}_{yy|1}}{\partial \tilde{x}} = 0.$$

Making allowance for the boundary conditions, we deduce that $\tilde{\tau}_1 = 0$ and $\tilde{\Sigma}_{yy|1} = \tilde{p}_0$. This means that, at least up to order 2, the expression of stress in a nonuniform regime is similar to that found in a steady regime. It is therefore possible to relate the bottom shear and normal stresses to the flow variables $(h,\ \bar{u})$. In most cases of interest, the normal stress along the flow direction Σ_{xx} may be written as

$$\Sigma_{xx} = -p + \mu\frac{\partial u}{\partial x}, \tag{2.34}$$

where μ is a viscosity coefficient. A dimensionless expression of this stress shows that it equals the pressure to leading order

$$\tilde{\Sigma}_{xx} = -\tilde{p} + \frac{\varepsilon}{\mathrm{Re}}\frac{\partial \tilde{u}}{\partial \tilde{x}}, \tag{2.35}$$

where $\mathrm{R}e = U_0H/\mu$ is a generalized Reynolds number. In absence of difference in normal stress, the normal stress Σ_{xx} may be approximated by a hydrostatic distribution.

A first application of the governing equations to nonuniform flows concerns the flow profile, namely the free surface profile (sometimes also called backwater curve in hydraulics) in a steady state. There are many reasons that motivate this examination. First, flow uniformity is a condition which is not systematically fulfilled in experiments. It is well known in hydraulics that the normal flow depth (i.e., the flow depth in a flow section where the uniform regime is achieved) is reached for steady flow taking place in sufficiently long channels. Conversely, in the case of short channels, the flow depth varies uniformly from the entrance to the exit without the normal depth being reached. Under these conditions, it may be rather delicate to evaluate the discharge equation. In practice, after ensuring the existence of the normal depth, it is possible to evaluate the discharge equation $q = F(h, \theta)$ and apply the reduction method [equation (2.22)] in order to find the steady-state constitutive relationship between the shear stress and shear rate. The reduction in flow depth in a finite size channel (downwards and upwards from this uniform section) may be also worked out to get information on normal stress. From equation (2.32) with $\partial\mathbf{S}/\partial t = 0$, we directly deduce that for a steady state

$$\nabla\mathbf{F}\cdot\frac{\partial \mathbf{S}}{\partial x} = \mathbf{G}^*(\mathbf{S}) \tag{2.36}$$

After rearranging the terms, we eventually find

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\tau_p - \bar{\rho}hg\sin\theta}{\alpha\bar{\rho}\bar{u}^2 + \frac{\partial h\bar{\Sigma}_{xx}}{\partial h} - \frac{\bar{u}}{h}\frac{\partial h\bar{\Sigma}_{xx}}{\partial \bar{u}}}. \tag{2.37}$$

Naturally, we find that, for uniform flows, we have: $\tau_p = \bar{\rho}hg\sin\theta$, which corresponds to the expression which can be deduced from equation (2.16). From the discharge equation, it is possible to deduce the relationship between the bottom shear stress and the flow variables $(h, \bar{u})$ in a steady state. In order to express the intrinsic character of this relationship (i.e., it is independent of the channel slope), we have to eliminate the inclination in the discharge equation $q = F(h, \theta)$ by setting: $\sin\theta = \tau_p/(\rho g h)$. By solving the resulting equation, we eventually obtain the bottom shear stress expression: $\tau_p = J(h, \bar{u})$. This constitutive relationship holds true for a steady state but, within the framework of long wave approximation, it is possible to use it in the gradually varying flow section upward or downward from the uniform section. Thereby, using this equation and equation (2.37), we find that

$$\frac{\partial h\bar{\Sigma}_{xx}}{\partial h} - \frac{\bar{u}}{h}\frac{\partial h\bar{\Sigma}_{xx}}{\partial \bar{u}} = \frac{J(\bar{h}, u) - \bar{\rho} h g \sin\theta}{h'(x)} - \alpha\bar{\rho}\bar{u}^2. \tag{2.38}$$

By measuring the (longitudinal) free-surface profile and estimating its local derivative, it is possible to get information on the variation of the normal stress $\bar{\Sigma}_{xx}$. As this equation does not give an accurate estimate of this quantity but only a quantity linked to it, it is mainly useful as a test for evaluating a given constitutive equation. Coupled to the transverse free-surface profile [equation (2.5)], it can provide a practical tool to get insight into the normal components of the stress tensor, which are often hard to measure by means of classical rheometers.

In most cases, the normal stress is negative (compression) and accordingly the denominator can vanish. This occurs for Froude number $(Fr = \bar{u}/\sqrt{gh})$ equal to a critical value

$$Fr^2_{c,\ s} = \frac{\bar{u}}{\alpha\bar{\rho} g h^2}\frac{\partial h\bar{\Sigma}_{xx}}{\partial \bar{u}} - \frac{1}{\alpha\bar{\rho} g h}\frac{\partial h\bar{\Sigma}_{xx}}{\partial h}, \tag{2.39}$$

where the subscript s indicates that this equation holds *a priori* for a steady state only. In absence of normal stress difference, the mean normal stress $\bar{\Sigma}_{xx}$ equals the mean normal stress $\bar{\Sigma}_{yy}$ and thus is independent of $\bar{u}$. It is straightforward to deduce that the critical Froude number is found to be equal to $\sqrt{\cos\theta/\alpha}$. In these circumstances, at a point where the denominator vanishes, the first derivative of the flow depth tends towards infinity; in other words, the flow profile should be normal to the flow direction at this point. The sudden change in the flow depth represents a discontinuity called a hydraulic jump in hydraulics. It should be noted that, at or near this critical point, the free surface is curved enough to cause significant deviations of the stream lines from the parallel flow plane. The governing equations (2.32) cannot be used to describe such a flow portion. A specific procedure must be used. An analysis similar to the one proposed by Savage (Savage 1979) may be performed to predict the depth change in a jump. Let us consider a control volume which includes both flow parts downstream and upstream of the granular jump (see Figure 2). The slope is assumed to be constant (even though in many practical cases, granular jumps are often caused by slope changes). The momentum equation applied to this volume yields

$$2\rho\bar{u}_1 h_1(\bar{u}_2 - \bar{u}_1) = P_1 - P_2 + W\sin\theta - L\tau_p, \tag{2.40}$$

where the subscripts i=1 and 2 refer respectively to the flow parts upstream and downstream of the jump, W denotes the weight of the control volume, and L its length, P_i $(i = 1,\ 2)$ are the mean normal forces $-h\bar{\Sigma}_{xx}$ acting on the flow sections. Similarly, the mass balance equation may be written as

$$\bar{u}_1 h_1 = \bar{u}_2 h_2. \tag{2.41}$$

Following Savage's idea, we may express the weight of the volume control per unit width

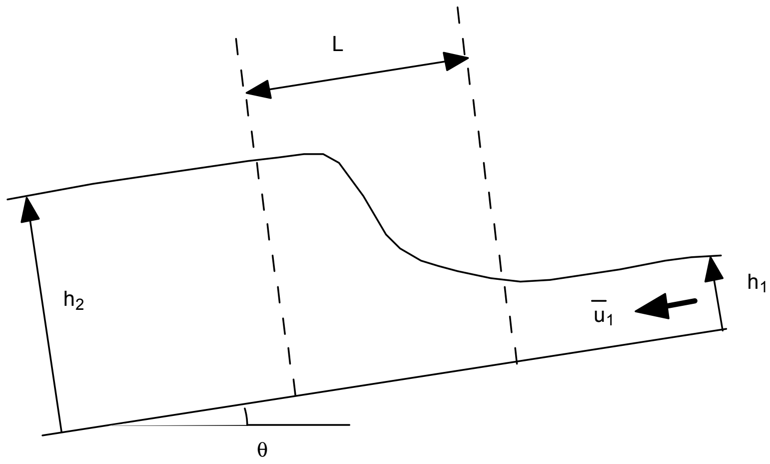

FIGURE 2. Sketch of a 'hydraulic' jump.

as:

$$W = \kappa \bar{\rho} g L \frac{h_1 + h_2}{2}, \tag{2.42}$$

where κ is a profile coefficient to account for the jump geometry ($\kappa = 1$ for a trapezoidal-shaped jump). Incorporating equations (2.41-2.42) into equation (2.40), we find that the normal stress $\bar{\Sigma}_{xx}$ satisfies the following equation (to leading order)

$$\frac{\bar{\Sigma}_{xx,1} - \xi \bar{\Sigma}_{xx,2}}{\bar{\rho} g h_1} = \kappa Z \frac{1+\xi}{2} \sin\theta - Z \frac{\tau_p}{\bar{\rho} g h_1} - 2 Fr_1^2 (\frac{1}{\xi} - 1), \tag{2.43}$$

where $\xi = h_2/h_1$ is the flow depth ratio, $Fr_1 = \sqrt{\bar{u}_1/gh_1}$ is the upstream Froude number, and $Z = L/h_1$. By measuring the various involved parameters $(Z, \theta, \kappa, Fr_1)$ and the flow depth ratio ξ, it is possible to estimate the variation of the normal component of the stress tensor $\bar{\Sigma}_{xx}$ with the flow depth. Conversely, if the normal stress is assumed to be known, it is possible to test the predictions of constitutive equations by solving the resulting polynomial and comparing the theoretical flow depth ratio to experimental data.

In hydraulics, the development of a jump pertains to the existence of two flow regimes: the supercritical ($Fr > 1$) and subcritical ($Fr < 1$) regimes. It is well known that specific properties are associated with each regime. Examples include the propagation direction of small gravity waves that occur in shallow water as a result of any momentary change in the local depth of the water: a gravity wave can be propagated upstream in water of subcritical flow but not in water of supercritical flow. Such a partitioning into flow regimes may be adapted to the present context. To that end, we are looking for the eigenvalues of the governing equations (2.32), which represent the velocity of small gravity waves. Using the Cayley-Hamilton theorem, we find that the eigenvalues are solutions to the following second order polynomial: $\lambda^2 - (tr\mathbf{G})\lambda + \det \mathbf{G} = 0$, whose discriminant is

$$D = 4\bar{u}^2\alpha + \frac{1}{\bar{\rho} h}\frac{\partial h \bar{\Sigma}_{xx}}{\partial \bar{u}} \left(\frac{1}{\bar{\rho} h}\frac{\partial h \bar{\Sigma}_{xx}}{\partial \bar{u}} - 4(\alpha - 1)\bar{u} \right) - 4\frac{1}{\bar{\rho}}\frac{\partial h \bar{\Sigma}_{xx}}{\partial h}. \tag{2.44}$$

We assume at this stage that D is always positive and therefore there are two eigenvalues

$$\frac{\lambda_{1,2}}{\sqrt{gh}} = \alpha Fr - \frac{1}{2\bar{\rho}\sqrt{gh^3}}\frac{\partial h \bar{\Sigma}_{xx}}{\partial \bar{u}} \pm \sqrt{\frac{D}{4gh}}. \tag{2.45}$$

In contrast to the Saint-Venant equations for which the eigenvalues are $\sqrt{gh}(F_r \pm 1)$ (Chow 1959), we found that in our case the eigenvalues are generally complex functions of the normal stress, except in the case where there is no normal stress difference. Depending on their sign, these small disturbances of the free surface may be propagated upstream or downstream. To specify the propagation direction, it is helpful to examine the existence of a critical Froude number corresponding to a vanishing eigenvalue ($\lambda = 0$). Equalling equation (2.45) to zero and rearranging the terms, we finally deduce

$$Fr_c^2 = Fr_{c,s}^2 \tag{2.46}$$

The critical Froude number for gradually varying flows equals the value found for the steady uniform flow. This result, holding true in hydraulics, is thus very general. In a similar way to the partitioning into fluvial and torrential regimes used in hydraulics, we can define

- a *subcritical* regime ($Fr < Fr_{c,s}$), where the two values have opposite signs. In this case the velocity of small perturbations may be lower than the mean velocity. Accordingly, the flow at a given point is controlled by both the downward and upward boundary conditions.
- a *supercritical* regime ($Fr > Fr_{c,s}$), where the two eigenvalues are positive. In this case, the velocity of small perturbations is larger than the mean flow velocity. Accordingly, the flow at a given point is controlled only by the upward boundary conditions.

The simple perturbation of the free surface with a pen and the subsequent observation of the disturbances can give information on the flow regime. If it is experimentally possible to find the critical Froude number separating supercritical from subcritical flows, then information on the form of the constitutive equation may be brought. For instance, if the critical Froude number is equal to $\sqrt{\cos\theta/\alpha}$, then there is likely no normal stress differences.

We can further draw the analogy with hydraulics to assess the stability domain of gravity-driven flows. It is well-known in hydraulics that the non-linear advection terms in the equations of motion cause the development of instability. For free surface flows down steep channels, this instability takes the form of small disturbances evolving towards successive steep waves separated by bores, which are called roll waves. As the structure of equations of motion studied here is similar to the shallow water equations, it is of interest to evaluate the conditions for which a flow (down an inclined channel) is stable. The present study differs from previous studies on the same subject, which examined stability using a perturbation analysis based on the local equations of motion. Here we prefer to use linear stability analysis of the flow-depth averaged equations (2.32) in a similar way to the global approach followed by Trowbridge (1987) for various types of fluids or Savage (1989) for granular flows. As the constitutive equations studied here may significantly differ from that of simple fluids, the instability criterion obtained by Trowbridge cannot be applied in the present context. We shall adapt Trowbrige's procedure to account for complex fluid specificities (notably effect of normal stresses). We assume that a steady uniform flow takes place, in other words there is a unique set (H, U) of normal flow depth and mean flow velocity values, which satisfies the equations of motion (2.32). We now consider small perturbations about the above solutions

$$\mathbf{S} = \mathbf{S}_0 + \mathbf{S}', \tag{2.47}$$

with the normal solution $\mathbf{S}_0 = (H, U)$ and the vector $\mathbf{S}' = (\eta, \kappa)$, where κ and η are respectively small disturbances of the mean velocity and flow depth. Substituting these terms in the equations of motion (2.32) and only keeping first-order terms, we obtain the following linearized equations for the perturbation quantities

$$\frac{\partial \mathbf{S}'}{\partial t} + \mathbf{G}(\mathbf{S}_0)\frac{\partial \mathbf{S}'}{\partial x} = \mathbf{G}^*(\mathbf{S}'). \tag{2.48}$$

Here we will examine the stability of modes of the following form

$$\eta = \mathrm{Re}(\Delta e^{i(nx-ct)}),\ \kappa = \mathrm{Re}(\mathrm{X}e^{i(nx-ct)}), \tag{2.49}$$

where Δ and X are the complex amplitudes respectively of flow depth and discharge, n is the wave number (which is a real positive number), and c a complex constant to be determined. The symbol i denotes the imaginary unity. The real part of c may be interpreted as the propagation velocity of the spatially periodic disturbances and its imaginary part reflects the growth (or decay) rate of the disturbance amplitude. Within the linear stability framework (Drazin & Reid 1981), the flow is considered as unstable as soon as it is possible to exhibit a solution to the system (2.48), for which the imaginary part of c is positive. Substituting the complex forms (2.49) into (2.48) yields a linear equation linear system

$$\left[\begin{array}{cc} nG_{11}-c & nG_{12} \\ nG_{21}-i\frac{\partial(\tau_p/\bar\rho H)}{\partial H} & nG_{22}-c-i\frac{\partial(\tau_p/\bar\rho H)}{\partial U}\end{array}\right]\left[\begin{array}{c}\Delta\\ X\end{array}\right] = \mathbf{0}, \tag{2.50}$$

where G_{ij} denotes the (i,j) component of the matrix $\mathbf{G}$ [see equation (2.32)]. This system has no trivial roots provided that its determinant is zero. Computing the determinant and equalling it to zero, we obtain the dispersion equation that we can put into the form of a second order polynomial in c

$$c^2 - 2\alpha c - \beta = 0, \tag{2.51}$$

with

$$\alpha = \alpha_r + i\alpha_i = n\frac{G_{22}+G_{11}}{2} - i\frac{1}{2}\frac{\partial(\tau_p/\bar\rho H)}{\partial U},$$

$$\beta = \beta_r + i\beta_i = n\left[n\left(G_{12}G_{21}-G_{22}G_{11}\right) + i\left(G_{11}\frac{\partial(\tau_p/\bar\rho H)}{\partial U} - G_{12}\frac{\partial(\tau_p/\bar\rho H)}{\partial H}\right)\right].$$

We are now looking for a solution to (2.51) rearranged in the form

$$(c-\alpha)^2 = re^{i\Theta}. \tag{2.52}$$

The imaginary part of the solution to Eq. (2.52) may be written as

$$c = \alpha \pm \sqrt{r}e^{i\Theta/2} \Rightarrow c_i = \mathrm{Im}(c) = \alpha_i \pm \sqrt{r}\sin\frac{\Theta}{2}. \tag{2.53}$$

The largest imaginary part is

$$c_i = \alpha_i + \sqrt{r}\left|\sin\frac{\Theta}{2}\right|. \tag{2.54}$$

We are searching the domain in which this number takes positive values: $c_i > 0$. By taking the square of the two sides of this inequality, then taking into account that $2\alpha_i^2 + r\cos\Theta$ is always positive and after rearranging the terms, we obtain:

$$r > 2\alpha_i^2 + r\cos\Theta \Leftrightarrow \beta_i^2 > 4\alpha_i(\beta_r\alpha_i - \beta_i\alpha_r). \tag{2.55}$$

We eventually deduce the instability criterion:

$$\begin{array}{l}\left(H\frac{\partial(\tau_p/\bar{\rho}H)}{\partial H}\right)^2+\left(2UH(\alpha-1)-\frac{1}{\bar{\rho}}\frac{\partial H\bar{\Sigma}_{xx}}{\partial U}\right)\frac{\partial(\tau_p/\bar{\rho}H)}{\partial U}\frac{\partial(\tau_p/\bar{\rho}H)}{\partial H}>\\\left((\alpha-1)U^2-\frac{1}{\bar{\rho}}\frac{\partial H\bar{\Sigma}_{xx}}{\partial H}\right)\left(\frac{\partial(\tau_p/\bar{\rho}H)}{\partial U}\right)^2\end{array}. \tag{2.56}$$

In many cases, when there is no normal stress difference, this condition may be simplified into:

$$\left(H\frac{\partial\tau_p}{\partial H}-\tau_p\right)\left(H\frac{\partial\tau_p}{\partial H}-\tau_p+2U(\alpha-1)\frac{\partial\tau_p}{\partial U}\right)>\left((\alpha-1)U^2+gH\cos\theta\right)\left(\frac{\partial\tau_p}{\partial U}\right)^2 \tag{2.57}$$

It may be shown that the source of energy for instability is work done on the perturbed flow by gravity and that the flow is linearly unstable if the rate of working by gravity exceeds the rate of energy dissipation by boundary friction (Trowbridge 1987). Equation (2.56) or (2.57) can hardly serve to get insight into the constitutive equation, but it is very useful to test the consistency of a constitutive equation by comparing the theoretical domain of stability and the limits estimated from experimental data.

3. Application to a simple case: the newtonian constitutive equation

For a Newtonian fluid, the constitutive equation is written: $\mathbf{\Sigma}=-p\mathbf{1}+2\mu\mathbf{D}$. Since the stress tensor is a linear function of the strain-rate tensor, there is no normal stress differences. For a simple shear flow down an inclined channel, it is straightforward to deduce the velocity profile

$$u(y)=\frac{\sin\theta}{\mu}\rho g y\left(h-\frac{y}{2}\right). \tag{3.1}$$

The Boussinesq coefficient α is equal to 6/5. The discharge equation is found by a simple integration of equation (3.1): $q=\rho g\sin\theta h^3/(3\mu)$. Using equation (2.22) yields

$$\dot{\gamma}=\tau/\mu \tag{3.2}$$

which corresponds well to the Newtonian constitutive equation in a steady-state simple-shear flow. From the discharge, we deduce that: $\sin\theta=3\mu\,\bar{u}/(2\rho g h^2)$ and thus the bottom shear stress may be written as: $\tau_p=J(h,\bar{u})=3\mu\,\bar{u}/(2h)$. As there is no normal stress differences, the critical Froude number (separating the supercritical and subcritical regimes) is equal to $\sqrt{5\cos\theta/6}$. From the instability criterion given by equation (2.57), we find that Newtonian flows down infinite channel are unstable as soon as:

$$Fr>\sqrt{\frac{\cos\theta}{3}}\approx 0.577+O(\theta^2) \tag{3.3}$$

This is to be compared to the value obtained by Benjamin (1957) and Yih (1963) using a more rigorous analysis on the stability of local equations of motion for Newtonian fluids: $\sqrt{5/18}\approx 0.527$. It is worth noticing that the critical Froude number pertaining to the loss of stability is lower than the Froude number corresponding to the appearance of the supercritical regime. This means that for laminar Newtonian fluids, no hydraulic jump may be observed.

4. Conclusions

In this paper, emphasis has been given on the relationship between bulk behavior and local properties of flows down inclined channels. Besides the determination of the constitutive equation in a steady-state simple-shear flow $\tau = f(\dot{\gamma})$, channel experiments are very helpful to infer another rheological parameters such as the first normal stress difference. Such experiments may be seen as discriminating since they can serve to test constitutive equations proposed in the literature or to get information on the form of the constitutive equation suitable for the studied materiel as well. For instance, the form of the transverse free-surface profile or the value of the critical number separating the supercritical and subcritical regimes can provide evidence on the existence of normal stress differences.

In comparison with other laboratory rheometers, a channel-type channel possesses many advantages, which make it very suitable for testing various kinds of materials. For instance, for suspensions of coarse particles, parallel plate rheometers or Couette cylinder are not always appropriate due to their limited gap with respect to the particle size. Fracture, depletion, migration, etc. are disturbing effects which may significantly affect rheometrical measurements. In contrast, even though these effects are still existing, their influence is often less pronounced. The basic principle of rheometry is to achieve (as much as possible) viscometric flows (i.e. flows in which the stress and velocity distributions are imposed). Here, only the stress distribution for the steady uniform regime is known whatever the type of constitutive equation. Information is taken both from uniform and nonuniform flows.

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