Similarity solutions for a gravity current in the high Reynolds-number limit for the shallow-water equations

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Gravity currents and geophysical flows

Boussinesq and non-Boussinesq flows
In the laboratory

Spread of a dense fluid (blend of aluminium particles and dense fluid)

⇝ Two large vortices

⇝ wedged-shape front

Shallow water (Saint-Venant) equations

\[
\frac{\partial h}{\partial t} + \frac{\partial h\bar{u}}{\partial x} = 0,
\]

\[
\frac{\partial \bar{u}}{\partial t} + (2\gamma - 1)\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{u}^2 \frac{\partial \gamma}{\partial x} = -\frac{\partial h}{\partial x} \left(1 + \frac{\bar{u}^2}{h}(\gamma - 1)\right).
\]

Similarity forms

\[u = \delta \xi \xi^{\delta - 1} V(\xi), \quad h = \delta^2 \xi^2 t^{2(\delta - 1)} Z(\xi), \text{ and } \xi = \frac{x}{t^\delta},\]

Boundary conditions

- At the front \(Z(\xi_f) = 0\) and \(V(\xi_f) = 1\) (non-Boussinesq regime).
- At the front \(Z(\xi_f) = Fr_f^2\) and \(V(\xi_f) = 1\) (Boussinesq regime: Benjamin condition).
- At the entrance:

\[Z \propto \beta \frac{1}{\delta^2 \xi^2} \text{ and } V \propto \beta^{3/2} \alpha \frac{1}{\delta \xi} \text{ when } \xi \to 0.\]
\[ M(V, Z) \frac{d\mathbf{w}}{d\xi} = \frac{Z}{\delta \xi} S(V, Z), \]

with \( \mathbf{w} = [Z, V]^T \),

\[ M = \begin{bmatrix} V - 1 & Z \\ (\gamma - 1)V^2 + Z & Z(V(2\gamma - 1) - 1) \end{bmatrix}, \]

and

\[ S = \begin{bmatrix} 3V\delta - 2 \\ 2\delta Z + V(V(4\gamma - 3)\delta - 1) \end{bmatrix}. \]

The determinant of the matrix \( M \) is \( \det M = \delta Z (Z - I(V)) \), with \( I(V) = 1 + (V - 2)V\gamma \).
Phase-plane formalism

Regular and critical points \((F = 0 \text{ and } G = 0)\)

\[
\frac{dZ}{dV} = \frac{F(V, Z)}{G(V, Z)}.
\]
Phase-plane for $\gamma > 1$

Problem: how to join S (source) and P (front point)?
Phase-plane for $\gamma = 1$

For $\gamma = 1$ the symmetry curve of the critical point (node) $A$ passes through $P$

$$\frac{dZ}{dV} = \frac{F(V, Z)}{G(V, Z)}$$

$$Z' = \frac{V_A F_V + Z_A F_Z + \cdots}{V_A G_V + Z_A G_Z + \cdots}$$
Solution for $\gamma > 1$

Jump between $A_\gamma$ and $A'$

Dashed line: approximate (i.e., to first order) analytical solution to the Euler equations
Conclusions

- A more physical construction of the solution in the tip region
- For non-Boussinesq regimes, subcritical similarity solutions do not exist.
- Supercritical similarity solutions exist for a limited range of volume growth $n$

\[
V = \int_0^{x_f} h(x, t) \, dx = At^n,
\]

with $1 \leq n < 2$. 