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Examination of the possibility of a fluid-mechanics treatment of dense granular flows

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SUMMARY

The aim of this paper is to examine the possibility of a simple fluid-mechanics treatment of rapid dense granular flows. In other words, we examine whether the constitutive equation can be sought in a simple relationship between the strain-rate and stress tensors. With this aim, we first show that an inclined channel is an appropriate device for providing rheological data. Here we provide a complete rheometrical treatment, which allows to infer the shear-stress/shear-rate curve (for simple shear flows) from the flow-depth/mass-flow-rate curve. Experiments performed with glass beads and sand grains revealed an apparent decrease in the shear stress with increasing shear rate. We then demonstrate that this result, although paradoxical, is not unphysical. Moreover, more detailed theoretical analysis shows that the main issues raised by our experiments may be overcome by 'microstructural' models. We finally give two examples of models including a single microstructural parameter, which are able to qualitatively account for the main features of our experiments.

KEY WORDS: granular flow; constitutive equation; simple fluid; friction; collision; microstructural model

1. INTRODUCTION

A large range of flows encountered in industry and nature involve highly concentrated mixtures of discrete solid grains and an interstitial fluid (generally water or air). By 'dense granular flow', we refer to any flow involving rapid and very large deformations of bulk solid mixtures, made up of non-colloidal particles and whose solid fraction (volume of solids per unit bulk volume) approaches the maximum packing concentration. Examples in the area of geophysics include rockfalls,[1] some rapid landslides and (stony) debris flows;[2] in addition, flowing avalanches are sometimes considered as granular flows.[3,4] In industry, we might quote some technological problems related to processing, transport, and handling of various materials (grains, cereals, sand, coal, pharmaceutical pills, ceramics, etc.).[5] An understanding of the dynamics of dense granular flows is of great interest in view of their practical importance, but unfortunately it is far from being complete despite numerous investigations. For instance, as far as we know, no theoretical model is able to correctly predict the macroscopic characteristics (such as depth/mass-flow-rate relations) of dense granular flows in various flow situations.

Here we examine the possibility of a fluid-mechanics treatment for dense granular flows. In other words, we shall attempt to determine whether a granular flow can be described using a simple constitutive equation, namely a relationship between the stress and strain-rate tensors. First, we

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intend to deduce experimentally the form of the constitutive equation. To this end, we show in the section 2 that an inclined open channel constitutes an appropriate device for studying dense granular flows and providing rheological data: we demonstrate how some components of the stress and strain-rate tensors can be derived as soon as the discharge equation (namely the relation between the flow rate and the flow depth) is established. In this demonstration, we restrict our attention to the case of isochoric flows. Section 3 presents the experimental device used for studying flows of glass beads or sand; some peculiarities of gravity-driven flows of particles are also mentioned. Section 4 discusses the results of our fluid-mechanics treatment and examines the assumptions underlying our experimental method. Finally, in the last section, we examine the possibility of a fluid-mechanics treatment from a theoretical point of view and in particular, we show that 'microstructural' models are able to account for our main experimental results.

2. AN INCLINED OPEN CHANNEL AS A RHEOMETRIC DEVICE

Here we shall consider that, for dense granular flows, the constitutive equation may be written in terms of a simple relationship between strain-rate and stress tensors. We rapidly recall the mathematical formulation of constitutive equations (for fluids) within the framework of rational mechanics. Then we show that for the *simple fluids*, some general properties of the constitutive equation can serve to directly deduce the equation from experiments.

2.1. General form of the constitutive equation for simple fluids

The central point of our fluid-mechanics treatment is that we assume the constitutive equation (for granular flows) to take the following simple form:

$$\Sigma = F(\mathbf{D}), \tag{1}$$

where $\mathbf{D}$ is the strain-rate tensor, Σ the stress tensor and F denotes an isotropic functional, which embodies the dependence of the stresses on the history of the relative deformation. Several crucial assumptions justify such a formulation:[6]

(i) the principles of determinism and local action can be applied;
(ii) the material is a continuum;
(iii) the material is homogeneous and isotropic.

The stress tensor may be written without restriction as the sum of a spherical tensor and an extra-stress tensor $\mathbf{T} = G(\mathbf{D})$:[6–8]

$$\Sigma = F(\mathbf{D}) = -p\mathbf{I} + \mathbf{T}, \tag{2}$$

where $\mathbf{I}$ denotes the identity tensor, p is a scalar quantity referred to as 'pressure' and G is the extra-stress (isotropic) functional.

Two complementary classes of materials can be represented by the relation (2).[6] The first class corresponds to compressible materials, for which the pressure is defined thermodynamically (using the free energy). The second class includes incompressible materials, for which the pressure is indeterminate and is found by solving motion equations. In this last case, to remove the non-uniqueness of G (due to the indeterminate pressure), the following convention is usual: $\mathrm{tr}\,\mathbf{T} = 0$.[7] In the following, we shall assume that any dense granular flow is isochoric.

2.2. Curvilinear flow down an inclined plane

The only information (but not the least important) that we can exploit from the simple generic form (2) is that Σ (or **T**) and **D** must be co-axial (i.e. both tensors have the same orthogonal eigenvectors). To pursue our investigations, we focus our attention on a peculiar class of flows. Let us consider a flow down an infinite plane inclined at an angle θ to the horizontal (see Figure 1). In practice, we shall see in the next section that the assumption of an infinite plane leads to a good approximation of flows down inclined channels of finite dimensions. Provided that this gravity-driven flow is uniform (i.e. the flow depth is constant) and steady (time is not an explicit variable), it is generally expected that the flow is *rectilinear* (in the reference text of Coleman *et al.*,[7] the authors preferred to use 'curvilinear'). In other words here, in the Cartesian coordinate system $R(x, y, z)$ of Figure 1, the components of the velocity field $\mathbf{v}(\mathbf{x})$ have the following form:

$$\mathrm{v}_x = u(y), \mathrm{v}_y = 0, \mathrm{v}_z = 0. \tag{3}$$

In addition, we assume that slip can occur at the bottom:

$$u(0) = u_g. \tag{4}$$

On account of the velocity field (3), the matrix of the components of the strain-rate tensor **D** may be written in the cartesian coordinate system R as follows:

$$D = \frac{1}{2}\dot{\gamma}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

where the shear rate $\dot{\gamma}$ may be expressed as a function of y and implicitly of the channel slope θ:

$$\dot{\gamma} = \left(\frac{\partial u}{\partial y}\right)_\theta. \tag{6}$$

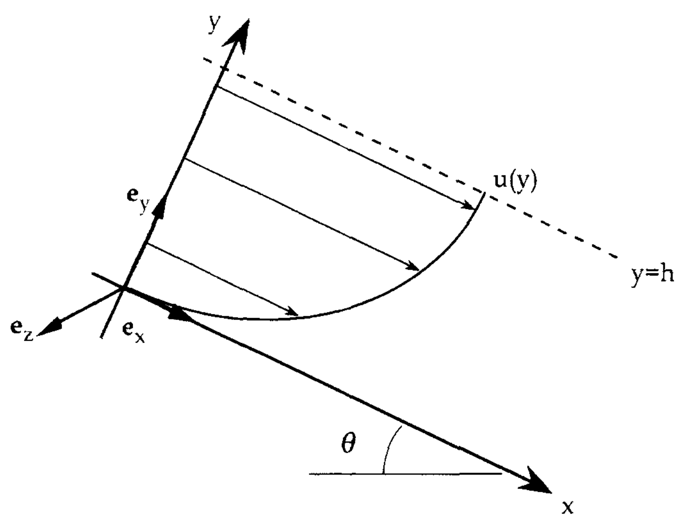

Figure 1. Notation and sketch of the flow down an infinite plane inclined at an angle θ to the horizontal

2.3. General form of stress-tensor components

We are looking for the simplest general form of the stress-tensor components for a curvilinear flow. In the coordinate system R, the components of the stress tensor may be written:

$$\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{xy} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{xz} & \Sigma_{yz} & \Sigma_{zz} \end{bmatrix} = \begin{bmatrix} \mathrm{T}_{xx} - p & \mathrm{T}_{xy} & \mathrm{T}_{xz} \\ \mathrm{T}_{xy} & \mathrm{T}_{yy} - p & \mathrm{T}_{yz} \\ \mathrm{T}_{xz} & \mathrm{T}_{yz} & \mathrm{T}_{zz} - p \end{bmatrix}. \tag{7}$$

Since the components of tensor $\mathbf{D}$ only depend on $\dot{\gamma}$, this holds true for the extra-stress-tensor components (owing to the definition (2)). Accordingly, we introduce the shear-stress function τ:

$$\Sigma_{xy} = \mathrm{T}_{xy} = \tau(\dot{\gamma}). \tag{8}$$

On the basis of stability considerations,[7,8] it may be shown that τ must be a continuous, positive, strictly increasing function of the shear rate and hence has an inverse, which we call the *shear-rate function*:

$$\dot{\gamma} = f(\tau). \tag{9}$$

Moreover, it may be shown that for a curvilinear flow, the stress tensor components must also verify:[7]

$$\Sigma_{yz} = \Sigma_{xz} = 0, \tag{10}$$

$$\Sigma_{xx} - \Sigma_{zz} = \sigma_1(\dot{\gamma}), \tag{11}$$

$$\Sigma_{yy} - \Sigma_{zz} = \sigma_2(\dot{\gamma}), \tag{12}$$

where σ_1 and σ_2 are only functions of the shear rate.

2.4. Examination of the momentum equations

The momentum balance may be written as

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho \mathbf{g} + \operatorname{div} \Sigma \tag{13}$$

where ρ and $\mathbf{g}$ denote the material density and the gravitational acceleration respectively. Since for steady rectilinear flows, the acceleration vanishes and the components of $\mathbf{T}$ depend on y alone, the motion equations reduce to:

$$0 = \frac{\partial \mathrm{T}_{xy}}{\partial y} - \frac{\partial p}{\partial x} + \rho g \sin\theta, \tag{14}$$

$$0 = \frac{\partial \mathrm{T}_{yy}}{\partial y} - \frac{\partial p}{\partial y} - \rho g \cos\theta, \tag{15}$$

$$0 = \frac{\partial p}{\partial z}. \tag{16}$$

It follows from (16) that p is independent of z, in other words, there is a function p such that:

$$p = p(x, y). \tag{17}$$

Accordingly, the equations (17) and (15) imply that p must write:

$$p(x, y) = T_{yy}(y) - \rho g y \cos\theta + a(x), \tag{18}$$

where $a(x)$ is a function of x. The equations (14) and (18) yield:

$$\frac{\partial}{\partial y}(T_{xy} + \rho g y \sin\theta) = \frac{\partial a}{\partial x}. \tag{19}$$

This is possible only if both terms of this equation are equal to a function of z, which we denote $b(z)$. Moreover, equation (16) implies that $b(z)$ is actually independent of z. Thus, in the following we shall note: $b(z) = b$. The solutions to (19) are:

$$a(x) = bx + c, \tag{20}$$

$$\mathrm{T}_{xy} + \rho g y \sin\theta = by + c', \tag{21}$$

where c and c' are some constants, which we shall determine.

To that end, let us consider the free surface: it is reasonable and usual to assume that air friction is negligible. Thereby, the stress continuity at the interface implies that the air pressure p_0 exerted on an elementary surface at $y = h$ (oriented by $\mathbf{e}_y$) must be equal to the stress exerted by the fluid. Henceforth, the boundary conditions at the free surface may be expressed as

$$-p_0\mathbf{e}_y = \Sigma\mathbf{e}_y, \tag{22}$$

which, using (7), implies that

$$T_{xy}(h) = 0, \tag{23}$$

$$p_0 = p(x, h) - T_{yy}(h). \tag{24}$$

Using (23), we obtain:

$$c' = -h(b - \rho g \sin\theta). \tag{25}$$

Likewise, using (18) (for $y = h$) and (24), we obtain:

$$a(x) = p_0 + \rho g h \cos\theta. \tag{26}$$

Accordingly, $a(x)$ is independent of x (in other words $b = 0$) and making use of (21) and (25), one obtains:

$$T_{xy} = \rho g \sin\theta(h - y). \tag{27}$$

It is worth noticing that the shear stress has been completely determined without knowing the constitutive relation. In the following, we shall use this property to derive the form of the shear-rate function from the discharge equation (relation between the flow rate and the flow depth).

2.5. Derivation of the constitutive equation

With this view, let us consider the flow rate (per unit width); it is defined as a function of h and θ:

$$q_v(h, \theta) = \int_0^h u(y)\,\mathrm{d}y. \tag{28}$$

An integration by parts leads to

$$q_v(h, \theta) = [(y - h)u(y)]_0^h + \int_0^h (h - y)\left(\frac{\partial u}{\partial y}\right)_\theta \mathrm{d}y. \tag{29}$$

In this equality, the first term of the right-hand term is hu_g owing to the slip condition at the bottom (4). Making use of (6), (9) and (27), (29) gives

$$q_v(h,\theta) = hu_g + \int_0^h (h-y) f(\rho g \sin\theta(h-y))\,\mathrm{d}y. \tag{30}$$

By making the variable change: $\xi = h - y$, we also obtain

$$q_v(h,\theta) = \int_0^h \xi f(\rho g \sin\theta\xi)\,\mathrm{d}\xi + hu_g. \tag{31}$$

Thus the partial derivative of q_v with respect to h (at a given channel slope θ) is

$$\left(\frac{\partial q_v}{\partial h}\right)_\theta = hf(\rho g h \sin\theta) + u_g + h\left(\frac{\partial u_g}{\partial h}\right)_\theta. \tag{32}$$

We call τ_p the shear stress at the bottom ($y = 0$):

$$\tau_\mathrm{p} = \rho g h \sin\theta. \tag{33}$$

The equation (32) is equivalent to

$$f(\tau_\mathrm{p}) = \frac{1}{h}\left(\frac{\partial q_v}{\partial h}\right)_\theta - \frac{u_g}{h} - \left(\frac{\partial u_g}{\partial h}\right)_\theta. \tag{34}$$

In the case (often encountered) of no-slip, this expression reduces to

$$f(\tau_\mathrm{p}) = \frac{1}{h}\left(\frac{\partial q_v}{\partial h}\right)_\theta. \tag{35}$$

This equation means that for a rectilinear flow down an inclined open channel, we are able to deduce the local behaviour from the bulk behaviour, given here by the discharge equation. It is worth noticing that this expression is very general and holds for any isochoric flow (down an open inclined channel) of materials, whose constitutive equation is in the form given by (1). Application of (34) amounts to using an inclined channel as a rheometer (see also Reference 9; this method has been already used for kaolin–water mixtures, and was successfully compared with independent rheometrical tests[10]).

3. EXPERIMENTAL RESULTS

3.1. Experimental apparatus and procedure

Experiments were performed with glass beads and sand grains (see characteristics in Table I). For the tests, we used a 2 m long and 2.5 cm wide PVC channel, fed via a hopper located at the upper channel entrance in a similar way to Johnson[11] or Patton.[12] The channel slope (θ) ranged from 0 to 40°.

Two macroscopic quantities can readily be controlled and measured: the mass flow rate per unit width (q) and the flow depth (h). The mass flow rate q was controlled by the hopper gate and ranged from 0 to 7.2 kg/s/m. The flow depth h was measured by ultrasonic sensors (Weidmüller LRS 3) using a scanner (Keythley 199) with a 50 Hz sampling rate. The maximum uncertainties on mass flow rate, flow depth and slope measurements were respectively 4%, 1% and 0.5%. Within the range of our tests, the flows are assumed to be isochoric, and accordingly, one can simply link the mass flow rate (per unit width) q and the flow rate (per unit width) q_v by: $q = \rho q_v = \nu\rho_\mathrm{p} q_v$, where ρ denotes the bulk density, ν the solid fraction, and ρ_p the particle density. Here, the solid-fraction value is taken as

Table I. Chief mechanical characteristics of particles used. We used samples of Hostun sand, which is often used in soil-mechanics experiments in France; its characteristics are given in References 82 and 83. The internal friction angle of both materials was determined by way of a triaxial. For each sample, we performed three drained, non-consolidated tests at a compression rate of 0.4 mm/mn. The value of ϕ was computed from the stress limit (perfectly plastic friction angle). The value of ϕ indicated in Table I is the mean value of three measures performed with various initial density.

	Glass	Sand
Mean diameter, d (mm)	0.3	0.32
d_{60}/d_{10}	1.2	1.6
Internal friction angle, ϕ (°)	27	36
Particle density, ρ_P (kg/m^3)	2460	2650
Particle shape	spherical	rounded

the mean value of solid fractions found for random loose packing of uniform beads:[13] $\nu = 0.6$. In the following, dimensionless variables will be used: mass flow rate by unit width (Q), depth (H), shear stress (T) and shear rate (Γ) which are respectively defined as: $Q = q/\rho d\sqrt{gd}$, $H = h/d$, $T = \tau/\rho g d$, and $\Gamma = \dot{\gamma}\sqrt{d/g}$, where g denotes the gravity, d the particle diameter.

We paid special attention to the roughness of the channel bottom. Indeed, previous experimental work has shown that the flow structure depends a great deal on the roughness of the bottom: when the bottom is smooth, slip clearly occurs and it is not sure that steady uniform flow exists within a large range of channel slopes.[11,14,15] On the contrary, if the bottom is rough enough, no slip occurs and steady uniform flows can take place over a large range of channel slopes.[15–18] Between this two limiting cases, there seems to exist a wide range of possible phenomena, including slip, torque transmission, etc., as shown in numerical simulations[19,20] and experiments on shear surface boundaries;[21] this can explain the occurrence of slip for rough surfaces as indicated by Johnson *et al.*[11] or Ahn *et al.*[22] In our case, the roughness was obtained by gluing beads, of which diameter was close to the mean particle diameter, onto the surface. Within the range of our experiments, no slip has been observed through the transparent sidewalls. In addition, we have also tested various roughness types. As shown in Figure 2, in the case of steady uniform flows, the flow depth was slightly affected by the roughness type as soon as the mass flow rate exceeded a critical value Q_c (see also below).

The experimental procedure consisted in measuring the flow depth for a given slope and a given mass flow rate once a steady state was reached. As already reported in the literature,[11,17] a steady uniform flow is possible over a wide range of mass flow rates for a channel slope in excess of the internal friction angle ϕ (see the sketch of Figure 3). However, for slightly more gentle slopes, steady flows (called *'immature sliding flows'* by Savage[23]) take place within a narrow range of mass flow rates while for larger mass flow rates, a stationary wedge-shaped layer develops along the channel bottom. When the channel slope becomes too steep, the free surface of the flow (called *'splashing flow'*[23]) is diffuse and is characterized by saltation of upper particles. We focused our attention on achieving of steady uniform regimes: the flow depth was measured at several places along the channel. For channel slopes larger than the internal friction angle, we found that the lengthwise profile flow depth varied, but there was still a fairly long part (at least 1 m) for which the flow depth was uniform. Except for this part, changes in flow depth were due to the finite size of the channel (boundary effect), as with gradually varied flows commonly encountered in hydraulics.[24] We also

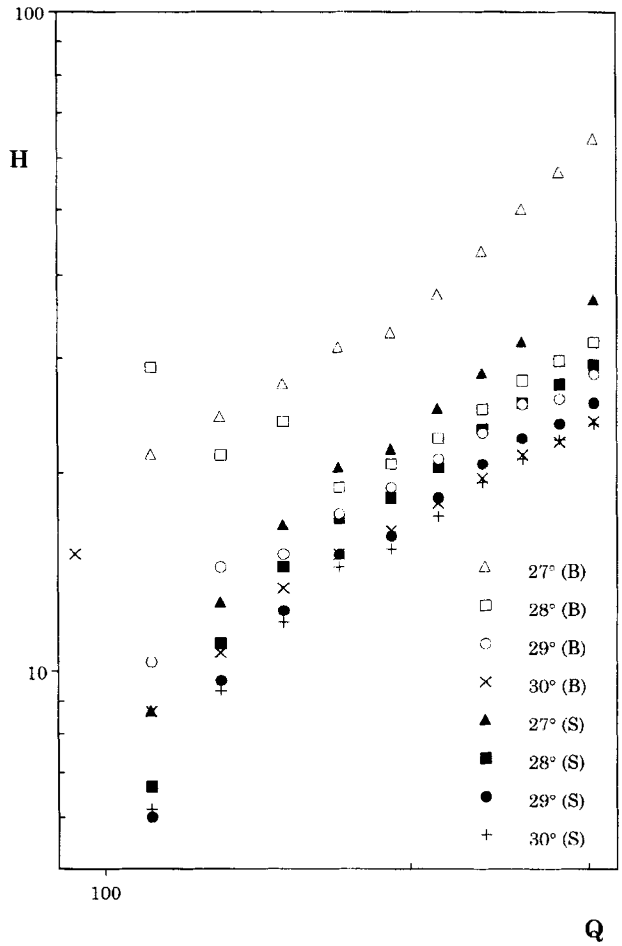

Figure 2. Influence of the roughness for two particular cases: for glass bead flows, we plot the dimensionless flow depth H as a function of the dimensionless mass flow rate Q for four channel slopes: $\theta = 27°, 28°, 29°, 30°$ and two roughness types. The former (referred to as 'B') was obtained by gluing beads of the same diameter onto the surface. The latter (referred to as 'S') was a sand paper made up of glued particles of diameter 0.2 mm. The critical mass flow rate Q_c is also indicated. For steady uniform flows (namely for $\theta > 27°$) and for $Q > Q_c$, the maximum relative difference between curves is about 8% (but reaches 200% when $Q < Q_c$)

observed that the length of the part of uniform depth significantly decreased with increasing slopes (in agreement with Reference 25).

3.2. Experimental results

For both materials, all couples (Q, H) corresponding to steady uniform flows are reported on Figures 4 and 5. For glass beads, data related to *immature sliding flows* are also presented to illustrate the flow pattern transition when the channel slope is decreased below the internal friction angle. Results for glass beads and sand grains were similar except that, for sand, the free surface exhibited significant fluctuations (up to 15% of the mean flow depth). Measurements were disturbed by this effect, which is probably due to particle shape, size distribution and shear-induced segregation. In the following, we shall only analyze typical data obtained with glass beads.

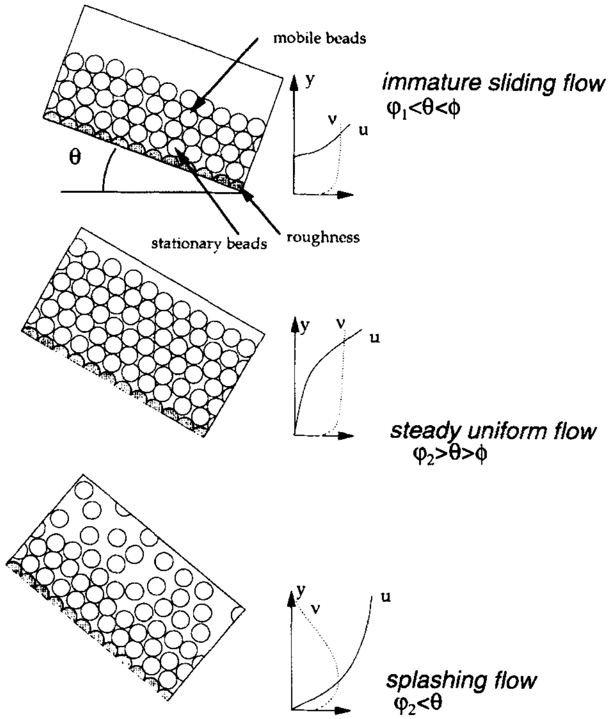

Figure 3. Sketch of observed flow patterns. Velocity profiles (full line) and density profiles (discontinuous line) are also roughly reported. The type of the flow regime is primarily controlled by the value of the channel inclination θ with respect to the internal friction angle ϕ.[18,23] Here, φ_1 and φ_2 respectively denote the angle of flow start (no motion is possible below this angle) and the angle of saltation start; contrary to ϕ, φ_1 and φ_2 are not intrinsic values of the material but depend on its mechanical characteristics and boundary conditions

As shown in Figure 4, the discharge curves exhibit a notable feature: there is a critical value of the (dimensionless) mass flow rate Q_c, which marks a significant change in the behaviour of steady uniform flows. This is particularly obvious for the extreme values of the range of tested sloped: for instance, the flow curve corresponding to the slope $\theta = 28°$ is characterized by a minimum at Q_c while the one corresponding to $\theta = 35°$ is characterized by a drastic jump. It is worth noting that the critical mass flow rate does not depend on the bottom roughness within our range of experiments and that the influence of the roughness becomes minor provided that the flow rate exceeds the critical value Q_c as shown in Figure 2. Concerning the discharge equation, we find that the flow depth is a linear function of the mass flow rate as soon as the mass flow rate exceeds the critical value Q_c:

$$Q = AH \sin^5 \theta. \tag{36}$$

For glass beads used in our tests, we find $A = 390 \pm 10$. The constant A is also found to slightly depend on the roughness. This experimental result is in complete agreement with the scant data published in the literature.[16,26] Moreover, from the discharge equation (36), we deduce that the mean velocity (defined as Q/H) only depends on channel slope. This would be natural if the flow were composed of a plug sliding on a shear band near the channel bottom. But, this is not the case since direct observations (through the transparent sidewalls) showed that within the range of our experiments, the material was sheared over its whole depth.

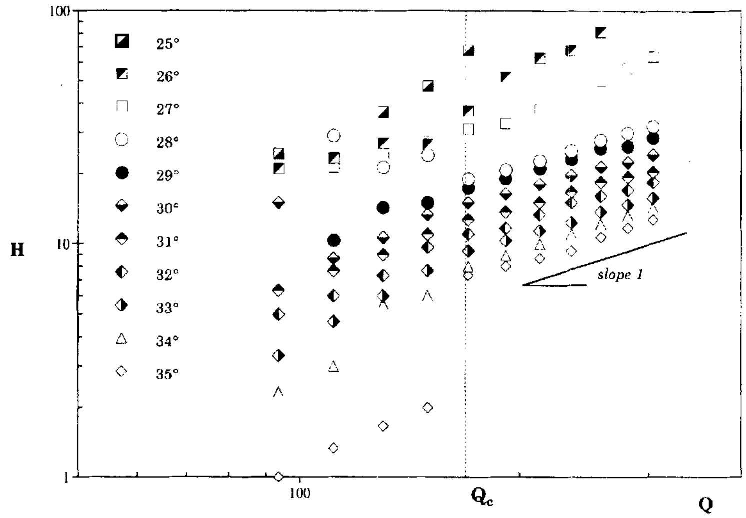

Figure 4. For the glass bead sample, dimensionless depth H is reported as a function of (dimensionless) mass flow rate Q for various channel slopes θ. For steady uniform flows, the discharge curves have a similar asymptotic behaviour (H varies linearly with Q). This asymptotic regime seems to take place as soon as the mass flow rate exceeds a critical value (Q_c), which is independent of the channel slope. This is particularly noticeable for $\theta = 28°$ and 35°. Here, we found $Q_c = 150 \pm 10$

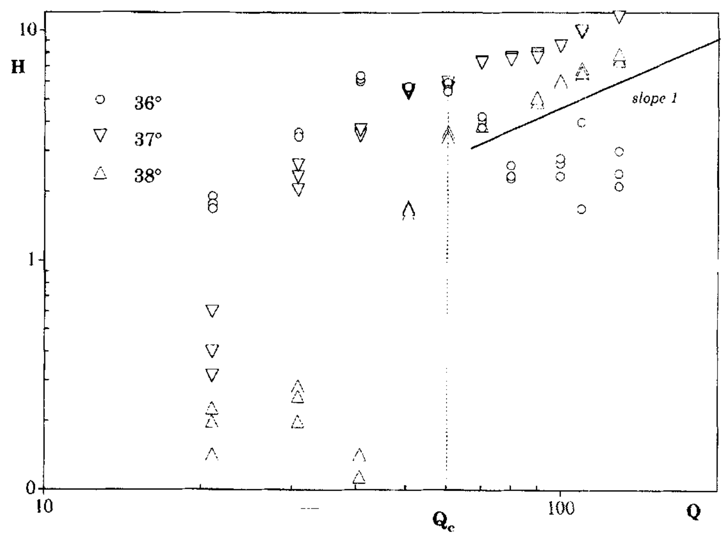

Figure 5. For the sand sample, non-dimensional depth H is reported as a function of mass flow rate Q for various channel slopes θ. For sand, the reported values correspond to averages on several depth measurements. The value of Q_c is found using the same remark as previously for the glass bead sample: $Q_c = 80 \pm 10$

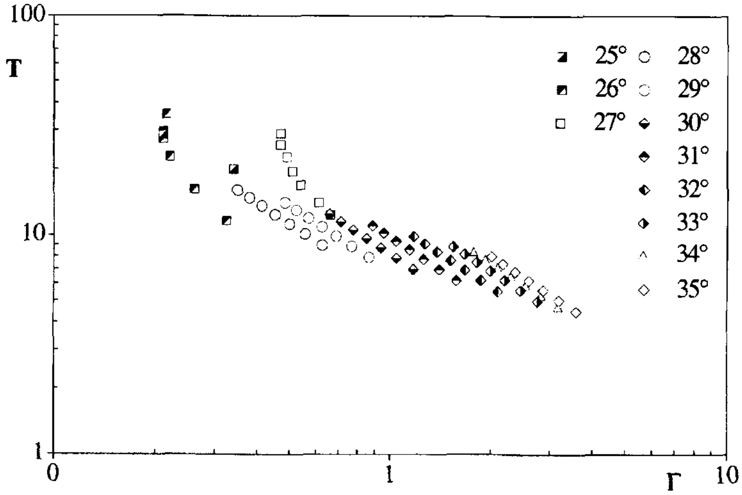

Figure 6. For glass beads, shear stress is plotted as a function of shear rate computed using (34) with experimental data (see Figure 3) for different channel slopes. Each set of data is related to its corresponding slope θ. Slopes 25° to 27° stand for immature sliding flows. For slopes $\theta > 28°$, data represent steady uniform flows

3.3. Derivation of the constitutive equation

In our context, the relation (36) can directly serve to derive the constitutive equation for glass beads. Making use of (33), (35) and (36) leads to

$$\Gamma = A \sin^5 \theta \frac{1}{H} = \frac{A \sin^6 \theta}{T}. \tag{37}$$

The formula (37) is applied for the data couples (Q, H) corresponding to steady uniform flows. The deduced points (Γ, T) are reported on Figure 6 in the form of a rheogram (namely shear stress versus shear rate).

4. DISCUSSION

4.1. Analysis of the rheogram

Normally, it is expected that all the points of the rheogram do collapse onto a single curve, which will stand for the constitutive equation in the case of a simple shear flow in a steady state. But obviously, this does not hold true in our case. Another noticeable fact is that the shear stress decreases for increasing shear rate. Theoretically, as mentioned above (section 2.3.), such a trend conflicts with the (linear) stability criterion of the curvilinear flows.[8,27] However, it is worth noting that this result is not unphysical. On the one hand, this trend is thermodynamically admissible since the stress power $P = \mathrm{tr}\,(\mathbf{D}\Sigma) = T\Gamma$ is still positive, in agreement with the second law of thermodynamics.[28] In addition, theoretical considerations and numerical simulations have already revealed an (apparent) decrease in the friction with increasing velocity.

For instance, the numerical model of Schmittbuhl *et al.* showed that the velocity dependence of the apparent friction is proportional to $1/V^2$ in the case of two rigid blocks with rough surfaces in relative motion the one to the other.[29]

Moreover, a decrease in the rheogram has already been observed for some materials, such as polymer melts[30] and concentrated suspensions or dispersions.[31–35] In this case, it has been sometimes argued that the existence of an apparent minimum in the flow curve is in fact due to wall slip or thixotropic effects.[30] Are such arguments possible in our context? Concerning the wall slip, we have paid special attention to obtain no-slip at the bottom wall within the range of our experiments (see section 4.2. also). The thixotropy concept is rather vague in the case of granular materials: generally, thixotropy is related to shear-history effects (for example, a reversible decrease in viscosity when a constant shear rate is applied to a sample at rest.[36]). As far as we know, the rapid granular flows do not exhibit properties typical of history-dependence and are not considered as thixotropic materials.

The behaviour of highly concentrated suspensions seems to be more closely related to granular flows: indeed, various experiments have shown that the flow curve seems to have a minimum as soon as the solid fraction approaches the maximum solid concentration.[34,35,37]

If relevant, these results give clear evidence that a fluid-mechanics treatment assuming a simple rheological model (in the form of (1)) is not sufficient for free-surface dense granular flows. It is interesting to review in more detail the assumptions on which the above method relies.

4.2. *Examination of the assumptions used in our treatment*

4.2.1. Assumption of isochoric flow. We have assumed that a dense granular flow is nearly isochoric, even if the granular materials are known as dilatant. Our observations (via video-films through the transparent sidewalls) crudely confirmed the results obtained by Patton *et al.*[12] Patton and co-workers used an ingenious system which involved trapping a flow portion. This showed that the mean bulk density was constant regardless of the mass flow rate, provided that the flow depth exceeded four particle diameters. This is also in agreement with the extensive experimental work done by Johnson *et al.*[11] together with Ishida and Hatano.[18] However, some authors have revealed significant variations in bulk density profile.[22,23,38] We suggest in this case that, although not explicitly stated, these authors studied immature sliding or splashing flows.

4.2.2. Assumption of infinite plane. Our analysis is based on the assumption that a flow running down a finite inclined channel can be treated as a flow along an infinite plane. This seems reasonable as long as the finite length effect and the sidewall influence are negligible. Concerning the former point, care has been paid to verify the existence of a portion of uniform depth over a significant length of the channel (about 1 m). This region is thought not to be disturbed by entrance and exit conditions. Concerning the latter point, our own observations of the free surface showed that significant slip occurred at the (smooth) sidewalls: using video films of the free surface, we estimated that (i) the slip velocity at the sidewall was roughly $2/3$ the velocity at the channel center, and (ii) the velocity profile across the channel width was significantly influenced by the walls only within a thin layer (the thickness of which was ten times the mean diameter of grains). This was in agreement with Savage's results on the surface velocity profiles.[16] We can point out that anyway, the sidewalls influence cannot explain the decrease in the flow curve. Both facts above confirm the validity of our assumption of infinite plane.

4.2.3. Assumption of no-slip at the channel bottom. The velocity at the channel bottom has been considered to be zero in accordance with our own observations and previous experimental works. However, several authors have emphasized the crucial role of boundary conditions for granular materials and have questioned its interpretation in a continuum mechanical treatment or the possibility of a no-slip condition:[19,20,39] for instance, in the Jenkins–Savage kinetic theory,[40] the no-slip condition led to an ill-posed boundary-value problem;[41] likewise the boundary conditions at the free surface raises some similar problems (see Reference 11 for instance). More generally, it is explained that in the immediate vicinity of a solid wall, several effects including solid-concentration decrease, torque transmission, impulse distribution, etc.,[19,20,39,42] strongly disturb the flow of particles. We can point out that the problem of boundary conditions has been addressed from a theoretical or numerical point of view, but not experimentally, except by Craig *et al.* for metal powders in an annular shear cell.[21]

Here, in our experimental context, we have performed several tests for two different bottom surfaces (see Figure 2). Within the range of our tests (for steady uniform flows), we have noted a slight influence of the roughness on the discharge equation (36) but no change in the form of (36) or in the flow pattern. This could mean that, although the particle velocity is zero at the channel bottom, it is necessary to introduce an apparent slip velocity in our rheometrical treatment (to account for the roughness-induced effects). To understand that, we give the example of concentrated suspensions:[43,44] in this case, the current practice consists in introducing an apparent slip (bulk) velocity, which is extrapolated from the (bulk) velocity profile measured far enough from the wall and which is different from the particle velocity in the vicinity of the wall. (This is tantamount to reducing the wall influence to a boundary layer, for an example of calculation procedure, see Reference 44.) However, even if we consider an apparent slip velocity at the bottom, it seems dubious that it can significantly affect the result of the rheometrical treatment: indeed, let us consider the ratio R of the contribution (to the shear function) due to the slip velocity on the contribution due to the mass flow rate in (34):

$$R = \frac{u_g}{(\partial q/\partial h)_\theta} \approx \frac{u_g}{A \sin^5 \theta}. \tag{38}$$

As u_g is not expected to exceed some cm/s, the ratio R is lower than 0.01, which is instrumental in proving that the approximation of no-slip is correct in our case.

4.2.4. Assumption of single-phase continuum. In our treatment, granular materials, have been considered as single-phase continua. Here it is risky to clearly justify this hypothesis and we shall restrict ourselves to give some arguments substantiating our treatment. We can point out that the existence of a free surface probably inhibits the creation of an air pressure gradient within the bulk flow and the main effect of air is thus due to viscous forces on the particle surface. The detailed picture of interstitial fluid effects should include various complicated hydrodynamics interactions[45–47] (lubricating contact, viscous dissipation due to collisions, etc.). None the less, we suggest that without a pressure gradient, the magnitude of hydrodynamic interactions may be given by Stokes' formula; this seems to be a fairly good approximation (see for instance the numerical simulations of Cichoki and Hinsen[48]). The relative importance of viscous forces on the gravity force using the following dimensionless ratio S is

$$S = \frac{\frac{4}{3}\pi R^3 \rho_p g}{6\pi R \mu_a V} = \frac{2}{9}\frac{g \rho_p R^2}{\mu_a V}, \tag{39}$$

where μ_a denotes the viscosity of air, V is a typical velocity of particles of density ρ_p. The number S is quite simply Stokes' number (see below). In our experiments, we found that S was greater than 60 and accordingly, this justifies the assumption of single-phase continuum. In addition, it can be expected that our experimental results would be sensitive to variations in atmospheric pressure. This effect has been observed for instance by Benarie with vane shear tests.[49] This can be readily explained in soil mechanics by considering the normal-stress dependence of the shear stress for a granular material under shear. In our case, it is expected that a change in air pressure only leads to a change in the magnitude of shear stress and accordingly does not imply any qualitative change of the flow pattern.

5. FURTHER THEORETICAL CONSIDERATIONS

5.1. Do the experimental results lead to a paradox?

Our simple experimental method questions the possibility of a fluid-mechanics treatment for dense granular flows. What conclusion should be drawn from this apparent failure? At first sight, this failure could reflect the shortcomings of the continuum approach to dense granulars flows. None the less, several arguments can be given in order to support this approach and justify that for dense granular materials, the formulation of a constitutive equation in the form $\Sigma = F(\mathbf{D})$ is meaningful. (For granular media, the continuum assumption is a postulate commonly used, even if some particular circumstances challenge the ability of continuum models to completely describe the granular behaviour: for instance, some systems exhibiting significant stress fluctuations.[50,51]) First, the bulk strain-rate and stress tensors can be successfully defined as averages over a set of realizations or over a representative volume (namely a volume including a sufficient number of particles). The consistency of such averaging procedures is experimentally[52–54] and theoretically[55–59] proved for a large range of media, whether they are (statistically) homogeneous or not. Secondly, it is possible (and usual) to treat granular media as single-phase media. Here, 'single-phase' means that one considers a mixture of particles and fluid as an equivalent continuous medium. This assumption is realistic as soon as the interplay between the fluid and solid phases is either very weak or very strong (for an introduction of problems related to two-phase flows, see Batchelor[60]); a convenient way to estimate interactions between both phases is to examine Stokes' number (as previously discussed in section 4.2.). Under the conditions above, the existence of a functional F linking the bulk tensors Σ and $\mathbf{D}$ is ensured.

The reason why our experimental treatment leads to a paradox is more probably due to the role played by some peculiarities of the granular microstructure, which we have so far ignored. In the following, we try to show that a microstructural approach to granular flows is able to overcome the main issues raised by our experimental results. Most of the current models are based on a microstructural approach inspired by the kinetic theory of gases,[23,42,61] (it should be noted that phenomenological models have been proposed[62–66] and generally lead to similar constitutive equations). The primary microstructural models were based only on momentum transfers by collisions. As agreement with experiments was far from being complete,[67,68] more recent models have included a frictional contribution in the constitutive equation. Briefly, from a formal point of view, we can summarize all the various expressions proposed for the constitutive equation by writing the stress tensor as the result of three separate mechanisms:[42,68–70]

$$\Sigma = \Sigma_c + \Sigma_k + \Sigma_f, \tag{40}$$

where Σ_c, Σ_k, and Σ_f, respectively, denote the collisional tensor (transport of momentum by collisions between particles), the kinetic tensor (transport of momentum due to the random 'thermal' motion of particles) and the frictional tensor (friction between particles at points of sustained contact). As the kinetic contribution is meant to rapidly decay when the solid fraction ν approaches the maximum packing concentration (0.5, say), it is usual to ignore it.[42] The collisional mechanism for granular materials is analogous to the molecular transport of momentum in liquids and is commonly expressed as a function of the strain-rate tensor, the solid fraction, the 'granular temperature' (related to the velocity fluctuations of the particles) and the radial distribution function (which can be considered as the distribution of angles of impact between particles).[41,42] At the macroscopic level, the frictional contribution gives a term similar to the yield stress in viscoplastic materials and it is usually independent of the rate of deformation. The yielding condition and the flow rule (which is assumed to be associated) are generally derived from the Coulomb law used in plasticity (for soil mechanics).[70] Contrary to the kinetic and collisional tensors, the frictional tensor is deduced empirically.

To sum up, in the current microstructural models for granular flows, there are three possible microstructrual parameters, which can control the transport of momentum in (40) and induce anisotropy in the material properties: the granular temperature, the radial distribution function, and the solid fraction. It should be noted that the nature of these scalar variables varies: the temperature is a kinetic parameter, the distribution function reflects the local geometric configuration of particles (and consequently gives information on the orientation of normal vectors at points of contact) and the solid fraction accounts for solid concentration. In the following, for convenience and making allowance for our experiments, we shall confine our attention to isochoric flows for which the solid fraction is assumed to be uniformly constant within the flow. This last assumption is prone to controversy, since it is sometimes argued that even very slight variations in solid fraction may lead to major changes in the stress magnitude. Anyway, even if this argument is relevant in our context, we think that the solid fraction cannot be retained conveniently as a control parameter and we prefer to resort to alternative microstructural parameters, which come closer to relate the variation in stress magnitude to the local configuration of particles (and indirectly account for changes in density). Moreover, owing to the considerable complexity of computation, most of the current models for granular flows have been obliged to disregard the anisotropy induced either by the granular temperature or by contact distribution. Here, we give two examples of models based on a single microstructural parameter, which, when applied to flows down inclined channels, lead to an apparent decrease in shear stress.

5.2. *Model with granular temperature as key-parameter*

The concept of granular temperature is often considered as the central ingredient in kinetic theories applied to granular flows.[42] Its introduction in rheological models (for granular flows) is largely motivated by the belief that the mean motion is controlled by the velocity fluctuations of particles. As the frictional-collisional models[11,68,70–72] generally lead to quite complicated equations of motion, which cannot be solved analytically, it is difficult to understand the actual role played by velocity fluctuations for particular flows.

We can point out that an explicit solution for dense granular flows can be obtained using the model of Johnson *et al.*,[11,71] subsequently simplified by Anderson and Jackson;[68] this model is a slightly modified form of the kinetic model proposed by Lun *et al.*[73] for nearly elastic, smooth, identical, and spherical particles; major modifications include the introduction of a frictional contribution (derived from the phenomenological Coulomb law) and the assumption that the 'thermal' motion of particles is negligible (in pseudo-thermal energy balance). Moreover, Johnson *et al.* have extended the work of Hui *et al.*[74] and Jenkins and Richman[75] on the boundary value problem. Using heuristical arguments similar to those used by Hui *et al.*, Johnson *et al.* derived the boundary conditions from force and energy balances for a volume of material in the vicinity of a solid wall; the particle–wall interactions embody friction and collisions. It is worth noticing that this formulation requires several 'specularity' parameters. Moreover, to ensure the consistency of their approach, the authors admitted that the bulk velocity was non-zero. In the simplified form of the Johnson's model, Anderson and Jackson assumed that the solid fraction was roughly constant over the depth and the bulk velocity gradient was zero (except near the bottom wall). This model provides a discharge equation in the form:

$$Q = BH^{3/2} \sin^{1/2} \theta, \tag{41}$$

where B is a function of the solid fraction, the channel inclination and bottom velocity. In addition, the bottom velocity is found to be a decreasing function of H:

$$U_g = \frac{C}{\sqrt{H}\tan\theta}, \tag{42}$$

where C is a function of the solid fraction. Making allowance for the asymptotic behaviour of the bottom velocity (for large flow depth), we prefer to ignore it in the following. Then, when using formula (35), equation (41) gives:

$$\mathrm{T} \approx \frac{9}{4}\frac{B^2 \sin^3\theta}{\Gamma^2}. \tag{43}$$

From a qualitative point of view, this model seems to be in agreement with our experiments: among others, the shear stress is found to decrease as the shear rate increases. In this case, it may be shown that the apparent decrease in shear stress is due to the variation in granular temperature over the depth.[68] When compared to our experimental relation (35), the discharge equation (40) presents some differences: it largely underestimates the influence of the channel slope whereas it slightly overestimates the influence of the flow depth. Moreover, the assumption of 'locked flow' (zero velocity-gradient) was not verified within our range of experiments.

From a quantitative point of view, no direct comparison with our own experiments were possible, since the model of Johnson *et al.* resorts to several phenomenological parameters, whose values probably must be fitted, but unfortunately, Johnson *et al.* did not indicate any sketch of their derivation.

5.3. *Model with non-uniform contact distribution*

Although in recent years, anisotropy in contact distribution has been shown by experimental,[76] numerical[77,78] and theoretical[79] results for a large range of shear flows, few constitutive equations for rapid granular flows account for it. As mentioned by Cheng[34] or Dent,[80] a possible alternative to frictional-collisional models based solely on granular temperature involves explaining that shearing causes a change in the arrangement of particles (layering effect), which in turn affects the distribution of contact points and implies a significant variation in stress. In this respect, in the case of gravity-driven flows down an inclined channel, we have recently proposed an *ad hoc* model: this model, based on momentum transfers via collisions and friction and allowing for contact angle anisotropy, accounts for the main features of our experimental results.[81] The shear-rate dependence of the shear stress is found to be

$$T = \frac{C\sin\theta}{\Gamma}\sqrt{\cos\theta(\tan\theta - \tan\phi)} \tag{43}$$

where C is a constant and ϕ the internal friction angle. The predictions of this simple model concerning the discharge equation are accurate to within 10% (for our data).

5.4. *Conclusion on frictional-collisional models*

It is worth noticing that both the above models, based on a constitutive equation in the form $\Sigma = F(\mathbf{D})$, result in a decreasing trend of shear stress, as the shear rate increases. Thus, they provide evidence that the paradoxical decrease in shear stress can be explained by constitutive equations which include a single parameter reflecting changes in the granular microstructure. The question whether these models are able to account for the actual rheological properties of granular properties is not clear: the simple fact that two models based on different assumptions yield similar results, is rather troublesome and might suggest that agreement with experiments would be fortuitous, or at least, it conceals a more complicated reality.

6. CONCLUDING REMARKS

Our experimental results have apparently questioned the relevance of a simple constitutive equation in the form $\Sigma = F(\mathbf{D})$ for dense granular flows. A possible way of explaining this paradox consists in introducing some variables reflecting the microscopic behaviour. Here we have given two examples of models which are able to predict an apparent decrease in shear stress: the former model emphasizes the granular temperature while the latter model is based on the non-uniformity of contact angles. None the less, qualitative agreement of both models with our tests does not completely ensure that these models based on a single scalar microstructural parameter reflects the actual rheological properties of granular flows.

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