A theoretical framework for granular suspensions in a steady simple shear flow

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Synopsis

We focus our attention on granular suspensions made up of noncolloidal spherical particles within a Newtonian fluid. The main objective of this paper is to provide a general framework for the formulation of the bulk stress tensor. The bulk stress within granular suspensions is mainly generated at the particle level by strong interactions between particles, such as friction, collision, and lubricated contact. The existence of a few local mechanisms is a major cause of behavior complexity at the macroscopic scale. A direct consequence is that the constitutive equation is only known for some flow conditions and given types of mixture. Here we have used a microstructural approach, which consists of considering the mixture as an effective continuum at the macroscopic level and inferring the bulk stress tensor from averaging of local interactions and local stresses. The bulk stress tensor may be split into elementary contributions pertaining to particle interactions. A complementary equation standing for the bulk energy dissipation may be needed in some circumstances. The analytical computation of these contributions is generally not possible. We present the various physical or heuristic reasonings usually proposed to circumvent this difficulty. © *1999 The Society of Rheology.* [S0148-6055(99)01606-5]

I. INTRODUCTION

A. Motivation and objectives

We focus our attention on highly concentrated mixtures of solid particles within a Newtonian fluid, which we call ‘‘granular suspensions’’ here. The particles are assumed to be large enough for the effects of Brownian motion and colloidal interactions to be negligible. A large number of natural, rapid, gravity-driven flows implies such mixtures. Examples include debris flows, avalanches, and rockslides. The constitutive equation of materials involved by such flows is known only for some given flow conditions that we shall specify below. The behavior of granular suspensions is governed by interactions

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occurring at the particle level. At high solid concentrations (the solid concentration is defined as the ratio of the solid volume to the total volume), these interactions are mainly due to strong interplay between solid particles. Three mechanisms are usually considered: friction, collision, and lubricated contact. The competition between these three interactions gives rise to a wide range of flow behaviors at the macroscopic scale. Initially, it is useful to only consider limiting flow regimes, for which a single type of interaction predominates.

In the case where the deformations are very small and fluid viscosity may be neglected, the flow is referred to as ''quasistatic'' flow. This regime has been studied widely in the context of soil mechanics. At the particle scale, stresses are transmitted through a network of frictional contact forces. At the macroscopic level, the behavior is typically plastic and may be correctly described by rate-independent frictional constitutive equations [Schofield and Wroth (1968), Vardoulakis and Sulem (1995)]. The role of the fluid phase is mainly limited to the fluid pressure in the pores; Terzaghi's postulate is an efficient way of breaking down the (total) bulk stresses into effective stresses (transmitted through the particle network) and interstitial pressure.

In the case where the deformation rates are slow enough and fluid viscosity plays a significant role, the flow regime has been called ''macroviscous'' by Bagnold (1954). This regime has been widely studied in the context of suspension rheology. At the particle scale, particle interactions are governed by hydrodynamic processes (such as lubricated contacts). At the macroscopic level, the behavior is usually described as Newtonian-like (in a simple shear flow), namely characterized by a linear relation between shear stress and shear rate.

In the case where the strain rates and the fluid viscosity may be neglected, the regime is most often referred to as ''grain-inertia'' regime according to Bagnold's classification (1954). At the particle scale, momentum exchanges are due to instantaneous particle collisions. Bulk stresses are found to be proportional to the square of the shear rate in a simple shear flow.

Apart from these limiting flow regimes, we have to consider two issues, which lag behind despite their great theoretical and practical importance. First, there is currently no practical way of delineating flow regimes. A part of the problem has been examined in the past. The first attempt is due to Bagnold (1954) who proposed a simple classification based on a single dimensionless number. Subsequent analyses by Krieger (1972), Savage (1984), Batchelor (1989) and Jomha (1990) have been based on a more detailed, but still partial dimensional analysis. A complete classification based on a series of dimensionless numbers should be very useful to predetermine the flow regime and thereby choose the convenient constitutive equation. Second, in many practical concerns, one is faced with transitional regimes, for which two or three interactions are competing. More attention must be paid to the development of reliable constitutive equations for transitional regimes, because they are not available or suffer a great deal from criticism.

In this paper, we propose an approximate physical scheme for establishing the regime classification and the constitutive equation of granular suspensions. To that end, several restrictions have been brought to simplify the problem. We only consider granular suspensions composed of identical, spherical, rigid, solid particles within an incompressible Newtonian fluid. The particles are assumed to be free of any external torque but submitted to a body force (assumed to be gravity here). In the following, we only consider simple shear flows, which can be regarded as laminar, isochoric, and one-phase flows at the macroscopic level (namely the mean velocities of solid and fluid phases are approximatively equal). Moreover, as a first approximation, it will also be assumed that there is a statistically uniform concentration of particles throughout the flow. We justify it by pointing out that the variations in the solid concentration due to shearing are low (they do not exceed a few percent). This might be considered as a dubious assumption since in most current views, granular suspensions are known to be dilatant materials and their bulk viscosity is found as a divergent function of the solid concentration. Accordingly, at high solid concentrations, minute changes should cause significant variations in bulk stress. But, in the meantime, particle arrangement is expected to influence, a great deal, bulk behavior. Data scattering exhibited by experiments may be seen as the result of inaccuracy in the concentration measurement or the influence of particle arrangement as well. From our point of view, for granular suspensions, a macroscopic parameter as the solid concentration must not be considered as a variable that controls the bulk behavior, but as the result of changes in the particle arrangement.

B. Presentation of the approach

Looking for the constitutive equation of granular suspensions is not a new subject. There is an enormous amount of literature presenting varied approaches developed to that end. On the whole, we can consider that two categories of approach have been developed during the last 30 yr.

Historically, the constitutive equation of granular suspensions has been mainly sought through refined tensorial expressions within the framework of continuum mechanics [Bedford and Dumheller (1983)]: the bulk stress tensor was expressed in polynomial form as a function of the strain-rate tensor and the concentration gradient. These models have been largely critized due to their poor concordance with experiments and some inconsistencies [Savage (1984)]. Great effort was then expended on proposing continuum models based on phenomenological rules [Berker and Van Arsdale (1992); Chen and Ling (1996)]. These models present the advantage of being computationally simple and efficient for the user. The drawbacks are that they are more or less arbitrary in their formulation and introduce variables that do not always have physical meaning and/or are not measurable (independently) at all.

Great advances have been made through the development of microstructural rheological models such as kinetic models for rapidly sheared regimes [Campbell (1990)] or homogenization techniques for quasistatic regimes [Emeriault *et al.* (1996); Rothenburg *et al.* (1989); Cambou *et al.* (1995)]. In these cases, the bulk stress tensor is inferred from averaging the particle behavior and momentum flux (at the microscopic level). These methods benefit from the fact that they are based on a well-established physical background. Nonetheless, the sophisticated mathematical formalism cannot hide the fact that these models are not able to establish a closed finite set of equations to calculate the constitutive equation; accordingly some additional hypotheses, numerical simulations or empirical relations are needed to circumvent this major issue. A typical example of such shortcomings is given by kinetic theories: in the first kinetic models, the radial distribution function g_0 (i.e., the probability of finding two particles at contact) has been deduced from the numerical results of Carnahan and Starling; subsequently, in order to come closer to experiments (notably to account for the expected viscosity divergence for solid concentrations ϕ close to the maximum concentration ϕ_m), this expression has been replaced by an empirical form which artificially introduces a divergence of g_0 for $\phi = \phi_m$. Consequently, difficulties in the development of these models derive from the fact that they are based on mechanical assumptions at the particle level, which are generally difficult to verify, and a limited number of local interactions that can be accounted for.

In this paper, we have focused on providing a unified view which attempts to reconcile the various concepts used in soil mechanics, physics of granular flow, and suspension rheology. We shall begin this paper with a rapid review of basic interactions within granular suspensions (Sec. II). On the basis of this survey, we then propose a series of dimensionless numbers which can contribute to obtaining a rough classification of flow regimes. In Sec. III, we tackle the rationale of the determination of bulk stress. First, we extend the method primarily employed by Batchelor to granular suspensions. In comparison to the work of Batchelor and subsequent developments, the originality of our approach lies in the assumption of rigid particles, which enables us to simplify a great deal the expressions concerning the solid phase. The bulk stress tensor may be calculated using appropriate averages on local stress. It might also be inferred from the rate of energy dissipation (per unit volume) of the bulk, but the equality between the two expressions of stress is not sure. On the contrary, we suggest considering the energy balance equation as a supplementary equation in a similar way to the thermal flux equation in kinetic theories or the fluctuating kinetic equation in turbulence. This also means that a mean-field approximation is not necessarily sufficient to treat granular suspensions. The main ingredient involved in computation of elementary contributions to bulk stress is the pair distribution function, namely the probability of finding two particles with given positions and characteristics. We review the main knowledge on this function for each regime and the methods used in the literature to infer the bulk stress tensor. In this last, part we focus our attention only on the computation of the particulate bulk stress tensor.

II. FLOW REGIMES

So far we have only provided a crude definition of a granular suspension. In this section, we will give a more detailed one by specifying the influence of the solid concentration on bulk behavior. For very concentrated suspensions, we provide evidence of the existence of a continuous network of particles in contact for a given range of solid concentrations, which gives specific properties to flows: dilatancy, layering, etc. (Sec. II A). This contact network explains why bulk behavior is governed by particle interactions. In the case of noncolloidal particles, we shall consider two types of contact (direct and indirect) depending on the presence of fluid film between particles (Sec. II B). In addition to particle–particle interactions, the momentum exchanges between solid and fluid phases may play an important role (Sec. II C). Owing to the scope of this paper, we shall restrict our attention to the cases where one phase is subordinated to the other. Finally, we propose a limited series of dimensionless numbers defined as the ratios of particle interaction magnitude. On the basis of this series, we identify different flow regimes corresponding to the predominance of one or two types of contact (Sec. II D).

A. Specificity of granular suspensions

Suspensions of solid particles within a Newtonian fluid exhibit various forms of behavior depending on the solid concentration. For dilute and moderately concentrated suspensions, particles are free to move through the bulk. For higher concentrations, motion of particles is increasingly impeded. When the solid concentration exceeds a critical value (ϕ_c), somewhat similar to a dynamic percolation yield [Blanc *et al.* (1983)], a continuous network of particles in contact forms throughout the bulk due to geometrical constraints. The formation of this network causes abrupt changes in flow behavior such as ordering of particles in layers (for simple shear flows), reordering of stress components, dilatancy, appearance of a minimum in the flow curve, and so on. Several experiments and numerical simulations on various systems provide clear evidence for these changes in close connection with the appearance of a contact network.

For instance, on the basis of several simulations on rapidly sheared flows of hard disks or spheres, Lun and Bent (1994) concluded that the critical concentration corresponds to the concentration of a face-centered cubic arrangement of particles: $\phi_c = \pi/6 \sim 0.52$ for spheres and $\phi_c = \pi/4 \sim 0.78$ for disks. Babic *et al.* (1990) employed a soft-disk model. They also concluded that there exists a critical solid concentration, but contrary to Lun and Bent, they found that it is not a fixed value: it depends to some extent on the shear rate and elastic characteristics of the material. For rigid disks, they again found that the critical concentration is given by $\phi_c = \pi/4$. Using Stokesian dynamics models, Brady and Bossis (1988) arrived at the same conclusion. Onoda and Liniger (1990) performed numerical simulations on the stability of piles and found that the critical concentration (that they called the random loose packing concentration) is close to 0.55. Experiments performed by Coussot (1997), Kytömaa and Prasad (1993), and Cheng (1994) on bead suspensions sheared in rheometers lead to values of ϕ_c close to 0.5. All these results show that there is not a fixed value of the critical solid concentration, but on the contrary, that it varies slightly around 0.52.

A second critical value of the solid concentration exists: it corresponds to the maximum random solid concentration, namely the concentration above which it is not possible to add new particles without bending them. In the case of random monodispersed mixtures, this concentration is: $\phi_m = 0.635$ [Meakin (1993)]. It follows that, for solid concentrations ranging from ϕ_c to ϕ_m, suspensions of solid particles are associated with a particle network throughout the bulk. We suggest calling such mixtures *granular suspensions*.

B. Particle interactions

For concentrated mixtures, the main dissipative processes are due to particle–particle interactions. Contact between two particles is a complex process, which generally depends on various basic mechanisms at the microscopic scale. To avoid an overly complicated description of tribological problems, we prefer to define the particle interactions through their effects and gross features at the particle scale. It is consequently very convenient to distinguish two main contact types: *direct contact* for which the particle surfaces meet (i.e., the distance separating the particle surfaces is equal to or less than the typical height of particle roughness) and *indirect contact* for which there exists a fluid film between particle surfaces.

1. Direct contact

In the following, we shall treat two limiting cases: *collisional contact* is referred to as a brief contact between two particles, whereas *frictional contact* is a sustained contact. Although these two forms of contact imply the same elementary physical mechanisms, their effects at the particle scale are very different. Owing to its very brief duration (with respect to a characteristic time of the flow), a collisional contact causes an exchange of momentum between two particles. As a consequence, the contact law is generally investigated in the form of a relation between the initial and final relative velocities (within the framework of rigid body mechanics). Conversely, the long duration of a sustained contact requires that a force (such as gravity) be applied to keep the two particles in contact. Thus the contact law is sought in the form of a relation between the forces transmitted during contact.

The simplest configuration of a collisional contact is the colinear collision between two spheres without initial spin. In this case, the collisional law takes the form of the well-known Newton law linking the pre- and postcollisional normal components of the relative velocity of mass centers, respectively c_n and c_n' [Johnson (1985)]:

$$c_n' = -ec_n, \tag{1}$$

where e denotes the coefficient of (normal) restitution; $e = 1$ when the collision only implies elastic deformations, and $0 < e < 1$ otherwise (in this case, the coefficient e depends on the precollisional velocity and the plasticity yield). The case of elastic oblique contacts or colinear impact between bodies with initial spin turns out to be more complicated. If the normal components of velocity can still be treated using Newton's law, the calculation of the tangential velocity component is intricate owing to the occurrence of stick–slip contacts. It may be shown that the postcollisional tangential component of the relative velocity (c_t') is a linear function of the precollisional velocity components (c_n and c_t) and the initial spin (Ω) [for instance see Foerster *et al.* (1994)]

$$c_t' = a_1 c_n + a_2 c_t + a_3 R\Omega, \tag{2}$$

where $R\Omega$ is the equivalent spin velocity given by $R\Omega = R_1\omega_1 + R_2\omega_2;\ \omega_1$ is the initial spin velocity of the bead of radius $R_i; a_i$ are three functions depending on the motion nature (sticking or slipping motion), and phenomenological material parameters. More complicated collision configurations, such as collision between three particles, are still an open question. There is some experimental evidence demonstrating that even Newton's law does not hold true for all collision configurations. For instance, in the case of a bead rolling on a bumpy line made up of juxtaposed beads, no rebound is experimentally observed. In this case, the failure of Newton's law is attributed to the propagation of elastic waves through the whole network of beads, but it may be shown that collisional-like momentum exchanges still occur and are the central mechanism of motion [Ancey *et al.* (1996)]. It can be expected that similar difficulties shall arise for dense rapid granular flows. For lack of a reliable formulation of contact law, we shall continue to use Newton's law to express momentum exchange between particles (as a first approximation).

For sustained contact between particles, frictional processes play a major role. Generally this contact is described using Coulomb's law, according to which two types of contact have to be distinguished depending on the slipping velocity at the point of contact: if this velocity is zero, contact is sticking and motion is referred to as *rolling without slipping;* otherwise, contact is slipping and motion is called *rolling with slipping*. For sticking contact, the tangential and normal components of the contact force (respectively S and N) satisfy: $S \leqslant \lambda N$, whereas for a slipping contact they satisfy

$$S = \lambda N. \tag{3}$$

Because of the scope of this paper, we accept that Coulombic friction is a fairly good description of frictional processes, even if it is now well established that various parameters (sliding velocity, temperature, oxydation film, etc.) can affect the value of the tangential force [Georges (1997)].

2. Indirect contacts

For noncolloidal particles within a Newtonian fluid, strong particle interactions can take place owing to lubrication processes. Contact is referred to as *indirect contact* or *lubricated contact*. The lubrication force between two spheres can be divided into three contributions: a squeezing contribution, a shearing contribution, and a term due to the rotation of spheres. It may be shown that, in a steady state, the squeezing contribution is to leading order [Cox (1974)]:

$$\mathbf{F}_{\mathrm{sq}} = \frac{3\pi}{2}\mu\frac{R^2}{\delta}\mathbf{c}_n, \tag{4}$$

where μ is the fluid viscosity and δ is the film thickness. The force due to shearing motion can be written to first order:

$$\mathbf{F}_{\mathrm{sh}} = \pi\mu R \ln\frac{\delta}{R}\mathbf{c}_t \tag{5}$$

and the force due to the rotation of particles is

$$\mathbf{F}_{\mathrm{rot}} = 2\pi\mu R^2 \ln\frac{\delta}{R}\mathbf{n}\times\mathbf{\Omega}, \tag{6}$$

where **n** is the normal joining the two mass centers. These two contributions are of the same order and their magnitudes increase as ln(δ/R). Consequently, for concentrated suspensions, to leading order in δ/R, they are negligible in comparison with the squeezing force. All the above expressions tend towards infinity when the gap becomes extremely small, which would preclude any direct contact. There exists naturally a minimum distance below which the expressions no longer hold. Although it is not the only mechanism justifying this minimum gap, we suggest here that it is due to surface roughness effects. In accordance with the experimental results obtained by Smart and Leighton (1993), we assume that the minimum gap is the order of the typical size of roughness (ε) and thus taking $\delta = \varepsilon$ in Eqs. (4)–(6) provides an approximation of the maximum lubrication forces.

C. Interplay between solid and fluid phases

Owing to the scope of this paper (derivation of constitutive equations), we must restrict our attention to flows of granular suspensions which may be regarded as laminar one-phase flows at the macroscopic scale. For a granular suspension to be considered as an equivalent continuum medium at the macroscopic scale, the mean velocities of each phase must be approximatively equal. Such a situation occurs when one phase is subordinated to the other or when the densities of the solid and fluid phases are approximatively equal. A practical way of evaluating phase interplay is to examine the motion of a single particle (of mass m, radius R and relative velocity **c**) submitted to the flow of a Newtonian fluid (of viscosity μ). In this case, the particle motion equation takes the form,

$$m\frac{d\mathbf{c}}{dt} = K\mathbf{c} + f(\mathbf{c},\Pi), \tag{7}$$

where K is a drag coefficient (in the case of Stokes flows, $K = 6\pi R\mu$), f is a function of the particle motion history and other parameters (denoted by Π) representing fluid effects, external force influence, etc. [Coimbra and Rangel (1998)]. Typically the force $K\mathbf{c}$ represents the chief action of the fluid phase on the particle. Let us define a characteristic time in connection with the fluid flow in the form: $t_f = \dot{\gamma}^{-1}$ and a characteristic time related to particle evolution: $t_p = m/K$. We can define a dimensionless number by taking the ratio of the two time values: $\mathrm{St} = t_p/t_f$. Let us examine the case where $\mathrm{St} \ll 1$. In this case, for variations over large time scales (namely of the order of t_f), we have $|dc/dt| \approx c/t_f$, which is negligible in comparison with c/t_p. Thus, according to Eq. (7), there exists a function h such that: $\mathbf{c} = h(\Pi)$. In this case, the particle is completely subordinated to fluid motion (the Stokes equation is applicable in a pseudo-static sense even though the flow is unsteady). Conversely, when the ratio St is large enough, the particle becomes independent of fluid variations.

In the case of Stokes flows, the dimensionless number St is naturally the Stokes number

$$\mathrm{St} = \frac{m\dot{\gamma}}{K} = \frac{m\dot{\gamma}}{6\pi\mu R} \approx \frac{\rho_p R^2 \dot{\gamma}}{\mu}, \tag{8}$$

where ρ_p is the particle density. For more complicated flows or when particles are added, the expression for St changes, but not the conclusion of the reasoning above. The question arises as to how the number St must vary for very concentrated suspensions. As far as we know, no simple answer exists and we propose the following rudiments of an answer. In the specific case of noncolloidal particle sedimentation, it has been shown that the settling velocity of a concentrated suspension may be empirically inferred by multiplying the settling velocity of a single particle by the factor $(1-\phi)^n$, with $n = 5.1$ [Davis and Acrivos (1985)]. By analogy with this problem, it may be expected that, for very concentrated suspensions, a reliable expression for K is $K = (1-\phi)^{-n} 6\pi\mu R$. Thus, the Stokes number given by Eq. (8) still reflects the ratio of characteristic times (apart from a multiplicative parameter depending on the solid concentration alone) and can be used to roughly qualify phase dependence whatever the solid concentration. When $\mathrm{St} \gg 1$, the fluid motion is mainly governed by particle movement and when $\mathrm{St} \ll 1$, the particle dynamics is chiefly dictated by hydrodynamics (Stokes flow) [Batchelor (1989); Russel *et al.* (1989)]. For intermediate values of St, the temporal acceleration of the fluid phase and particles must be taken into account.

D. Classification of flow regimes

Here we propose a simplified diagram of predominant interactions with granular suspensions. A more general framework is provided in a companion paper [Coussot and Ancey (1999a,b)]. To that purpose, we have to compare the magnitude of each interaction to the others. Thereby three dimensionless numbers may be defined as the ratios of magnitudes. For collisional interactions to be dominant, the particle inertia must be large enough to overcome the hydrodynamic repulsive force. This can be examined by considering the ratio of particle inertia (mc^2) to the work of the normal lubricated force [Eq. (4)] computed between a certain distance below which the lubrication force begins to be significant (ε^*) and the minimum distance (ε):

$$\mathrm{Ba} = \frac{4\pi}{3\ln(\varepsilon^*/\varepsilon)} \frac{\rho_p R^3 c^2}{\mu R^2 c} \approx \frac{1}{\ln(\varepsilon^*/\varepsilon)} \frac{\rho_p R^2 \dot{\gamma}}{\mu}, \tag{9}$$

where we used the following approximation for estimating the relative normal velocity: $c \approx 2R\dot{\gamma}$. We suggest calling it the Bagnold number; let us notice that from a dimensional point of view, it is similar to the Stokes number and the so-called particle Reynolds number. We can also compare the magnitude of collisional and frictional contacts. As there exists a contact network, the frictional force N acting on particles during sustained contact is of the order of the ‘‘effective’’ stress Σ (defined bulk stress minus fluid contribution in soil mechanics). We may define the following number, which we suggest to call the Coulomb number:

$$\mathrm{Co} = \frac{\rho_p R^2 \dot{\gamma}^2}{\Sigma}. \tag{10}$$

Likewise, we can compare the magnitude of frictional and lubricated forces. We propose calling the corresponding ratio the Leighton number (due to the similarity of the resuspension mechanism studied by Leighton and Acrivos)

$$\mathrm{Le} = \frac{\mu R \dot{\gamma}}{\Sigma \varepsilon}. \tag{11}$$

Naturally these dimensionless numbers are not independent since we have the following relations: $\mathrm{Le} \approx \mathrm{Co}\, R/(\varepsilon\, \mathrm{Ba})$. Using these numbers, we can delineate three regimes corresponding to the predominance of one type of contact:

(i) a frictional regime for $\mathrm{Co} \ll 1$ and $\mathrm{Le} \ll 1$,
(ii) a collisional regime for $\mathrm{Co} \gg 1$ and $\mathrm{Ba} \gg 1$,
(iii) a hydrodynamic regime for $\mathrm{Ba} \ll 1$ and $\mathrm{Le} \gg 1$.

Between these extreme regimes, there exist transitional regimes for which the bulk stresses are generated by the combination of two different kinds of particle interactions. The frictional–collisional regime is reached when frictional and collisional interactions are competing, namely for $\mathrm{Co} \approx 1$ and $\mathrm{Le} \ll 1$ (or $\mathrm{Ba} \gg 1$). The frictional–hydrodynamic regime occurs when bulk stress results from the combination of lubricated and frictional contacts, namely for $\mathrm{Le} \approx 1$ and $\mathrm{Co} \ll 1$ (or $\mathrm{Ba} \ll 1$). The hydrodynamic–collisional regime should correspond to the case where collisional and lubricated contacts are the main mechanisms of stress generation. As this case should be associated with $\mathrm{Ba} \approx 1$ or equivalently $\mathrm{St} \approx 1$ (two-phase flows), we can question the existence of such a regime in our context.

Naturally, due to the large number of parameters involved in the dynamics of granular suspensions, more dimensionless groups are required to fully specify the bulk behavior. For instance, the density ratio or the Davis number (reflecting particle rigidity versus hydrodynamic force magnitude) are dimensionless numbers, which appear in the motion equations when one addresses the issue of fluid–particle interactions [Coimbra and Rangel (1998)] or particle–particle interactions [Lian *et al.* (1996)] in the hydrodynamic regime. Alternative numbers, such as the Froude number, will also appear depending on the bulk flow context. Here, in order to avoid a flurry of dimensionless groups, it is suggested that the main trends of bulk behavior (or equivalently the flow regime) may be indicated with only three numbers: Stokes, Coulomb, and Bagnold. Inside each flow regime, it is necessary to use additional dimensionless numbers when detailing particular solutions to motion equations.

III. DERIVATION OF THE BULK STRESS TENSOR

A. Definition of bulk stress

In a fundamental paper, Batchelor (1970) has defined the bulk stress as the average of local stress plus a term pertaining to momentum flux:

$$\Sigma = \bar{\sigma} - \overline{\rho \mathbf{u}' \otimes \mathbf{u}'}. \tag{12}$$

In this expression, the overbar represents any appropriate average (since we shall assume ergodicity of the system and accordingly equivalence between average processes) and the prime indicates fluctuations around the mean value: $\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}}$. The definition of stress proposed by Batchelor has been critically examined by Zhang and Prosperetti (1997). These authors have focused on the calculation of momentum and energy equations for disperse two-phase flows. They derived the bulk stress for each phase of the mixture from

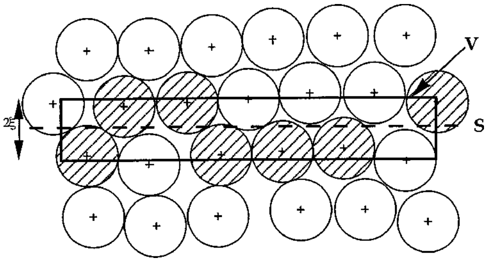

FIG. 1. Sketch of the integration volume *V* centered around the shear plane *S*. The dashed particles are particles whose mass center is included in the volume *V*.

the local stress. They also showed that their total stress is equivalent to Batchelor's definition of bulk stress, except that their expression includes a correction term accounting for nonuniformity in particle distribution.

In the following, we will mainly use volume and ensemble averages. To ensure the equivalence between these two averages in our context, we have to take for granted the ergodicity property. Here we provide no clear evidence for that, but some considerations can serve to support this idea [Batchelor (1970)]. A sufficient condition for ergodicity is to find a volume, with a length which is very large compared to the particle size *R* and over which the mixture is statistically homogeneous. To that end, let us consider a plane surface *S* parallel to the flow direction (see Fig. 1). This surface cuts through fluid and particles. We then consider a volume *V* centered on *S* and with a depth $\xi < R$. Within the volume *V*, the mixture is necessarily statistically homogeneous and hence, for this typical volume, we shall use both ensemble or volume averages. We may note that particles concerned by the volume *V* are not completely included in this volume. This will prove rather cumbersome in the subsequent treatment. Accordingly, we suggest modifying the shape of *V*. We have assumed that the volume *V* cuts a large number *N* of spheres. As the probability that a particle is beneath *S* is equal to the probability that it is above *S*, there is an equal chance that the mass center of the particle intersected by *S* is either inside *V* or outside *V*. We define a new volume V' from *V* by including any particle, the mass center of which is inside *V* and removing any particle, the mass center of which is outside *V* (see Fig. 2). The measurement of V' is equal to the one of *V*. In the following, for brevity, we shall use the notation *V* instead of V', but the reader must keep in mind that the integration volume is V'.

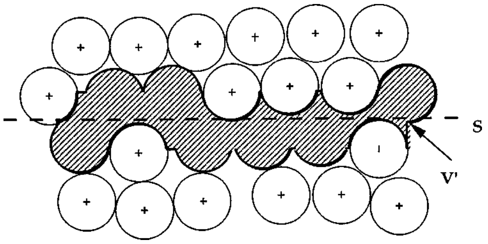

FIG. 2. Sketch of the integration volume V'.

B. Derivation of the volume-averaged bulk stress

First, we shall use a volume averaging method. As the fluid is assumed to be Newtonian, the (local) constitutive equation is: $\sigma^{(f)} = -p\mathbf{1}+2\mu\mathbf{d}$, where p is the fluid pressure, μ is the fluid viscosity, and $\mathbf{d}$ is the local deformation rate. Moreover, it has been assumed that the particles can be regarded as rigid bodies. It is henceforth possible: (i) to ignore the stress distribution inside particles and (ii) consider that the velocity distribution for rigid spheres is controlled by the velocity of the mass center $\mathbf{u}^{(n)}$ and angular velocity $\Omega^{(n)}$:

$$\mathbf{v}(M,t) = \mathbf{u}^{(n)}(t)+\mathbf{\Omega}^{(n)}(t)\times[\mathbf{x}(M)-\mathbf{x}^{(n)}] \tag{13}$$

for any point M inside the sphere (n) centered at $\mathbf{x}^{(n)}$. It is assumed that the fluid does not slip at the particle surface. The volume-averaged strain rate will be denoted $\bar{d}_{ij}$:

$$\bar{\mathbf{d}} = \frac{1}{V}\int_V \mathbf{d}dv = \frac{1}{V}\int_{V_f}\mathbf{d}dv. \tag{14}$$

In Eq. (14), V_f denotes the part of the volume occupied by the fluid. Likewise, the volume occupied by particles will be denoted V_p. In Eq. (14), we were able to ignore the particle contribution to the average strain rate due to the fact that the velocity gradient vanishes inside rigid particles. In terms of volume average, Eq. (12) is equivalent to

$$\Sigma = \frac{1}{V}\int_V(\sigma-\rho\mathbf{u}'\otimes\mathbf{u}')dV. \tag{15}$$

As the integration volume may be broken down into a fluid and solid subvolumes, we can write the bulk stress as the sum of a fluid contribution and a particle contribution

$$\Sigma = \Sigma^{(f)}+\Sigma^{(p)}. \tag{16}$$

As the fluid is assumed to be Newtonian, it is straightforward to show that the fluid contribution may be written as [Batchelor (1970)]:

$$\Sigma^{(f)} = 2\mu\bar{\mathbf{d}}-\frac{1}{V}\int_{V_f}pdV-\frac{1}{V}\int_{V_f}\rho_f\mathbf{u}'\otimes\mathbf{u}'dV. \tag{17}$$

The particle contribution expression requires more attention. It is

$$\Sigma^{(p)} = \frac{1}{V}\sum_{n=1}^{N}\int_{V_p^{(n)}}(\sigma-\rho_p\mathbf{u}'\otimes\mathbf{u}')dV, \tag{18}$$

where $V_p^{(n)}$ is the volume occupied by the nth sphere of V. Likewise $A_p^{(n)}$ will denote the surface of the nth sphere. When applying the divergence theorem in the first term of the right-hand expression, we find

$$\int_{V_p^{(n)}}\sigma dV = \int_{A_p^{(n)}}(\sigma\mathbf{x})\mathbf{n}\,dS-\int_{V_p^{(n)}}\mathbf{x}(\nabla\cdot\sigma)dV, \tag{19}$$

where n is the outward normal. We have to evaluate the second term on the right-hand side of Eq. (19), in which the divergence of the stress field intervenes. As the particles are considered as rigid this term is indeterminate under this form. To get round this difficulty, we attempt to link the stress field inside a particle to the forces exerted on this particle. Inside a particle, the momentum equation is

$$\nabla\cdot\sigma = \rho_p\left(\frac{\partial\mathbf{v}}{\partial t}-\mathbf{g}\right). \tag{20}$$

Acceleration may be calculated using Eq. (13)

$$\frac{\partial\mathbf{v}}{\partial t} = \frac{\partial\mathbf{u}^{(n)}}{\partial t}+\boldsymbol{\Omega}^{(n)^2}\mathbf{r}'+\dot{\boldsymbol{\Omega}}^{(n)}\times\mathbf{r}, \tag{21}$$

where we used: $\mathbf{r} = \mathbf{x}-\mathbf{x}^{(n)}$ and $\mathbf{r}' = \mathbf{r}-(\boldsymbol{\Omega}^{(n)}\cdot\mathbf{r}/\Omega^{(n)^2})\Omega^{(n)}$. We deduce

$$\frac{\partial\mathbf{v}}{\partial t} = \frac{\partial\mathbf{u}^{(n)}}{\partial t}+\boldsymbol{\Omega}^{(n)^2}\mathbf{r}'+\dot{\boldsymbol{\Omega}}^{(n)}\times\mathbf{r}. \tag{22}$$

The last step consists of linking the derivatives of the velocities to the external forces. To that end, let us consider the motion equation for the nth bead:

$$m_p\frac{d\mathbf{u}^{(n)}}{dt} = m_p\mathbf{g}+\mathbf{F}_{\text{ext}}^{(n)}, \tag{23}$$

where m_p denotes the particle mass and $\mathbf{F}_{\text{ext}}^{(n)}$ the resulting force exerted on the nth particle, defined as

$$\mathbf{F}_{\text{ext}}^{(n)} = \int_{A^{(n)}}\sigma\mathbf{n}\,dS. \tag{24}$$

The connection of the angular velocity to the resulting external force can be deduced using the angular momentum equation:

$$J_p\frac{d\boldsymbol{\Omega}^{(m)}}{dt} = \mathbf{M}_{\text{ext}}^{(m)} = \int_{A_p^{(m)}}\mathbf{r}\times(\sigma\mathbf{n})dS, \tag{25}$$

where J_p denotes the moment of inertia of the particle and $\mathbf{M}_{\text{ext}}^{(m)}$ the resulting torque exerted on the mth particle. From Eqs. (22), (23), and (25), we derive the new expression for Eq. (20):

$$\nabla\cdot\sigma = \rho_p\left(\frac{\mathbf{F}_{\text{ext}}^{(n)}}{m_p}+\Omega^{(n)^2}\mathbf{r}+\frac{\mathbf{M}_{\text{ext}}^{(n)}}{J_p}\otimes\mathbf{r}-\mathbf{r}\cdot\boldsymbol{\Omega}^{(n)}\boldsymbol{\Omega}^{(n)}\right). \tag{26}$$

Accordingly, on the basis of Eq. (26), it is straightforward to calculate the second term in the right-hand expression of (19):

$$\int_{V_p^{(n)}}\mathbf{x}\cdot\nabla\cdot\sigma\,dV = \frac{1}{2}J_p(\Omega^{(n)^2}\mathbf{1}-\Omega^{(n)}\otimes\Omega^{(n)})+\frac{1}{2}\mathcal{N}(\mathbf{M}_{\text{ext}}^{(n)}), \tag{27}$$

where we have introduced the following antisymmetric operator (skew product):

$$\mathbf{x}\overset{\mathcal{N}}{\rightarrow}\mathcal{N}(\mathbf{M}_{\text{ext}})\mathbf{x} = \mathbf{M}_{\text{ext}}\times\mathbf{x}. \tag{28}$$

It is now possible to express the particle contribution to the bulk stress. On account of Eqs. (18), (19), and (27), we can show that the particle contribution may be broken down into three contributions:

$$\boldsymbol{\Sigma}^{(p)} = \boldsymbol{\Sigma}_s^{(p)}+\boldsymbol{\Sigma}_a^{(p)}+\boldsymbol{\Sigma}_f^{(p)}, \tag{29}$$

where the first term represents the surface forces acting on individual particles

$$\boldsymbol{\Sigma}_s^{(p)} = \frac{1}{V}\sum_{m=1}^{N}\left[\int_{A_p^{(m)}}\sigma\cdot\mathbf{r}\otimes\mathbf{n}\,dS - \mathcal{N}\left(\frac{1}{2}\int_{A_p^{(m)}}\mathbf{r}\times(\sigma\cdot\mathbf{n})dS\right)\right]. \tag{30}$$

The second term reflects the influence of angular velocity:

$$\boldsymbol{\Sigma}_a^{(p)} = \frac{1}{2}\frac{1}{V}J_p\sum_{m=1}^{N}(\boldsymbol{\Omega}^{(m)}\otimes\boldsymbol{\Omega}^{(m)} - \Omega^{(m)^2}\mathbf{1}). \tag{31}$$

The third term pertains to velocity fluctuations:

$$\boldsymbol{\Sigma}_f^{(p)} = -\frac{1}{V}m_p\sum_{n=1}^{N}\mathbf{u}'^{(n)}\otimes\mathbf{u}'^{(n)}. \tag{32}$$

C. Derivation of the ensemble-averaged bulk stress

Further calculations on a volume-averaged bulk stress are not very helpful. At this stage, it is more convenient to use the ergodicity hypothesis to replace volume-averaged terms by ensemble-averaged terms. (We do not specify the definition of the ensemble average here and refer the readers to Sec. V and more detailed papers). For any continuous field X defined on the solid phase, we have the following relation between the volume and ensemble averages (provided that the volume includes a very large number of particles):

$$\frac{1}{V}\sum_{n=1}^{N}X(V_p^{(n)}) = n_d\langle X(V_p^{(0)})\rangle, \tag{33}$$

where $n_d = \phi/V_p$ is the density number and $V_p^{(0)}$ represents a test particle, and for any continuous field X defined on the fluid phase, we have (owing to the hypothesis of uniform concentration)

$$\frac{1}{V}\int_{V_f}X\,dV = (1-\phi)\langle X\rangle. \tag{34}$$

The particulate contributions, Eqs. (30)–(32), can now be written as

$$\boldsymbol{\Sigma}_s^{(p)} = Rn_d\langle\mathbf{f}\otimes\mathbf{n} - \tfrac{1}{2}\mathcal{N}(\mathbf{n}\times\mathbf{f})\rangle, \tag{35}$$

$$\boldsymbol{\Sigma}_a^{(p)} = \tfrac{1}{2}n_dJ_p(\langle\boldsymbol{\Omega}^{(0)}\otimes\boldsymbol{\Omega}^{(0)}\rangle' - \langle\boldsymbol{\Omega}^{(0)^2}\rangle\mathbf{1}), \tag{36}$$

$$\boldsymbol{\Sigma}_f^{(p)} = -n_dm_p\langle\mathbf{u}'^{(0)}\otimes\mathbf{u}'^{(0)}\rangle. \tag{37}$$

The fluid contribution is deduced from Eq. (17)

$$\Sigma^{(f)} = 2\mu\bar{\mathbf{d}} - (1-\phi)(\langle p\rangle\mathbf{1} + \langle\rho_f\mathbf{u}'\otimes\mathbf{u}'\rangle). \tag{38}$$

In Eq. (35), $\mathbf{f}$ represents the external force applied to an elementary surface dS of the particle surface: $\mathbf{f} = \sigma\cdot\mathbf{n}dS$ due to fluid or particle actions. Owing to its definition given in Eqs. (16) and (29), the resulting bulk stress is found by calculating the sum of Eqs. (35)–(38). We can point out that the form of the particulate stress expression slightly differs from the one established by Batchelor [Batchelor (1970)]:

$$\Sigma^{(p)} = Rn_d\langle\mathbf{f}\otimes\mathbf{n} - \mu(\mathbf{u}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{u}) - \tfrac{1}{2}\rho\mathbf{a}'\otimes\mathbf{n} - \rho\mathbf{u}'\otimes\mathbf{u}'\rangle, \tag{39}$$

where $\mathbf{a}'$ is the acceleration fluctuation of the particles. Indeed, Batchelor's expression for bulk stress holds for any suspension of force-free particles, whether rigid or not. [Here we adapt Batchelor' expression in terms of ensemble averages with the assumption that no external torque is applied on particles.] In our case, we have considered rigid particles submitted to gravity. A direct consequence is that: (i) we neglected particle deformation and the interface layer and used sticking and nonpenetrability of fluid at the particle surface to eliminate the terms in $\mathbf{u}\otimes\mathbf{n}$ in our expression and (ii) we linked the divergence of stress inside the particles to the global features of the motion equation of particles rather than the local acceleration fluctuations as done by Batchelor. Procedure (i) is usually employed in suspension theories as soon as one deals with rigid particles [Phan-Tien (1995); Van der Brule and Jongshaap (1991)]. Procedure (ii) has been used only by a few authors like Zhang and Prosperetti (1997), who defined a ''particle'' ensemble average different from the process used for the fluid phase. These authors do no include the particle angular velocities in their phase space and accordingly, a stress contribution like Eq. (36) does not exist in their expression of particulate stress. As a last remark, it is worth noticing that the particulate contribution, Eq. (35), encompasses an antisymmetric term.

IV. BULK ENERGY EQUATION

The energy conservation may be written for a volume V in the absence of a heat source:

$$\int_v \rho \frac{dE}{dt} dv = \int_v \rho \mathbf{g}\cdot\mathbf{u}\, dv + \int_{\delta v} \mathbf{u}\cdot(\sigma\mathbf{n}) dS, \tag{40}$$

where $E = \varepsilon + u^2/2$ is the total energy per unit volume, and ε the internal energy. Using the divergence theorem and the symmetry of the local stress tensor, we may also write

$$\int_v \rho \frac{dE}{dt} dv = \int_v \rho \mathbf{g}\cdot\mathbf{u}\, dv + \int_v \operatorname{div}(\sigma\mathbf{u}) dv - \sum_{m=1}^{N} \int_{A^{(m)}} \mathbf{n}\cdot]\sigma\mathbf{u}[dS, \tag{41}$$

where $]\sigma\mathbf{u}[$ denotes the jump of $\sigma\mathbf{u}$ through $A^{(m)}$ (due to direct or indirect contacts between close particles). Using the fact that the momentum equation implies

$$\mathbf{u}\cdot \operatorname{div}\sigma = \rho\mathbf{u}\cdot\left(\frac{d\mathbf{u}}{dt} - \mathbf{g}\right), \tag{42}$$

we deduce

$$\frac{d\bar{\epsilon}}{dt} = \frac{1}{V}\int_V \sigma:\mathbf{d}\, dv + \frac{1}{V}\sum_{m=1}^{N} \int_{A^{(m)}} \mathbf{n}\cdot]\sigma\mathbf{u}[dS. \tag{43}$$

After introducing the average of the strain rate and its fluctuating part $\mathbf{d}'$, we have

$$\frac{d\bar{\varepsilon}}{dt} = \frac{1}{V}\int_V \sigma:\bar{\mathbf{d}}\, dv + \frac{1}{V}\int_V \sigma:\bar{\mathbf{d}}'\, dv + \frac{1}{V}\sum_{m=1}^{N} \int_{A_{(m)}} \mathbf{n}\cdot]\sigma\mathbf{u}[dS \tag{44}$$

After rearranging the terms, we find

$$\frac{d\overline{\varepsilon}}{dt}=(\Sigma+\overline{\rho\mathbf{u}'\otimes\mathbf{u}'}):\overline{\mathbf{d}}+\frac{1}{V}\int_V[\nabla\cdot(\sigma\cdot\mathbf{u}')-(\nabla\cdot\sigma)\cdot\mathbf{u}']dv+\frac{1}{V}\sum_{m=1}^{N}\int_{A^{(m)}}\mathbf{n}\cdot]\sigma\mathbf{u}[dS. \tag{45}$$

Using the local momentum equation and rearranging it again leads to the following expression:

$$\frac{d\overline{\varepsilon}}{dt}+\frac{1}{2}\frac{d}{dt}\overline{\rho u'^2}=\Sigma:\overline{\mathbf{d}}+\nabla\cdot\mathbf{q}+\frac{1}{V}\sum_{m=1}^{N}\int_{A^{(m)}}\mathbf{n}\cdot]\sigma\mathbf{u}]dS, \tag{46}$$

where $q=\overline{\sigma\cdot\mathbf{u}'}$ is the stress flux due to velocity fluctuations. Expression (46) differs slightly from the one given by Batchelor (1970) in the absence of external torque acting on particles. The main difference lies in the fact that Eq. (46) accounts for dissipative solid contacts (sustained contact between solid particles), whereas Batchelor did not consider such a process. For suspensions of noninteracting particles or when contacts between particles are short, the third term in the right-hand side of Eq. (46) is negligible. We also kept the divergence of **q** which has been considered as negligible by Batchelor. Let us also note that when the dissipation term due to contacts is negligible, Eq. (46) is formally similar to the one used in kinetic theories for dry granular flows [Campbell (1990)].

The interpretation of Eq. (46) is the following: the apparent rate of dissipation (per unit volume), defined from bulk quantities $\Sigma:\overline{\mathbf{d}}$, is equal to the true rate of dissipation only if the rate of fluctuation kinetic energy and the rate of dissipation due to contacts can be neglected. In this case, the notion of an effective continuum model for the mixture is relevant; it should be tantamount to deriving the bulk stress tensor from averaging microscopic stress [Eq. (12)] and deducing it from the macroscopic rate of dissipation: $\sigma_{ij}=\partial\overline{\dot{E}}/\partial\overline{d}_{ij}$. Conversely, in the case where the contribution of dissipative contacts or fluctuation kinetic energy can no longer be neglected, these two ways of defining the bulk stress are not equivalent; another consequence is that, in our context, considering the bulk as a continuum can raise some issues (like the nonequivalence between the true and apparent rates of dissipation). An example of discrepancy is given later by the computations of the bulk viscosity for concentrated suspensions in the hydrodynamic regime (see Sec. V D). Here, to avoid any inconsistency within our treatment, we suggest considering the energy balance equation [Eq. (46)] as a supplementary equation, required to close the motion equations (balance of momentum and mass), in a method similar to the common practice in turbulence or kinetic theories. In these fields, the value of bulk viscosity is not self determined but depends on the fluctuating kinetic energy flux.

V. CONSTITUTIVE EQUATION FOR VERY CONCENTRATED MIXTURES

A. Presentation of the approach

In the last two sections, we have given the expressions for the bulk stress and the rate of dissipation. We are now concerned in determining the constitutive equation for granular suspensions on the basis of the equations above. The usual method for pursuing calculations in that direction is to employ appropriate ensemble averages [Batchelor (1972); Batchelor and Green (1972); Herczynski and Pienkowska (1980); Zhang and Prosperetti (1994)]. The ensemble average of any function $f(\mathbf{x},t;\mathcal{C}^N)$ is usually defined as

$$\langle f(\mathbf{x},t)\rangle = \frac{1}{N!}\int_{\mathcal{C}^N} P(t;N)f(\mathbf{x},t;\mathcal{C}^N)d\mathcal{C}^{\mathcal{N}}, \tag{47}$$

where $\mathcal{C}^N$ denotes the configuration of N particles (specified by their positions, linear and angular velocities) in the volume V and $P(t;N)$ is the probability of encountering the configuration $\mathcal{C}^N$ at time t. It should be noted that the degrees of freedom of the fluid phase should be included in the variables describing the dynamical state of the system (except for inviscid fluid or Stokes interstitial flow). Due to the extremely large amount of information contained in the probability density $P(t;N)$, it is convenient to use reduced distributions of lower orders. Using conditional averages, it follows from the definition, Eq. (47) [Herczynski and Pienkowska (1980)]:

$$\langle f(\mathbf{r},t)\rangle = \int_{\mathcal{C}^2} P_2(t;\mathbf{x},\mathbf{y})\hat{f}^{(2)}(\mathbf{x},t;\mathcal{C}^N)d\mathbf{x}\,d\mathbf{y}, \tag{48}$$

where P_2 is the *pair distribution function* defined as the probability that simultaneously the centers of two spheres lie, respectively, in $\mathbf{x}$ and $\mathbf{y}$:

$$P_2(t;\mathbf{x},\mathbf{y}) = \frac{1}{N(N-1)}\int P(t;\mathbf{x},\mathbf{y},\mathcal{C}^{(N-2)})d\mathcal{C}^{(N-2)}, \tag{49}$$

where $\mathcal{C}^{(N-2)}$ denotes the remaining $N-2$ particles. Likewise, $\hat{f}^{(2)}$ denotes the conditional averaged function when the position of two spheres is fixed:

$$\hat{f}^{(2)} = \int_{\mathcal{C}^2} P(t;N-2|\mathbf{x},\mathbf{y})f(\mathbf{r},t;\mathcal{C}^N)d\mathcal{C}^{(N-2)}, \tag{50}$$

where the conditional probability $P(N-2|\mathbf{x},\mathbf{y})$ is the distribution probability of the remaining $N-2$ spheres when two spheres are fixed at $\mathbf{x}$ and $\mathbf{y}$: $P(N-2|\mathbf{x},\mathbf{y}) = P(N)/P_2(\mathbf{x},\mathbf{y})$. As suggested by Batchelor (1972), (1974) in a kinetic theory or for dilute suspension rheology, it is expected that the function is mainly affected by the particles close to the reference particle rather than distant ones. Therefore it is assumed that the conditional averaged function $\hat{f}^{(2)}$ may be merely replaced by f. Apart from systems governed by fluctuations (supercritical phase transition), such an assumption is generally sound. For dilute suspensions, using simple conservation principles, it is generally possible to infer the differential equation governing the variations of the probability P and thereby, to compute it for some particular flow conditions. Depending on the system, such an equation is known as a Boltzmann equation, a Smoluchowki equation, etc. Its generic expression is

$$\frac{\partial P}{\partial t} + \nabla\cdot\mathbf{F}(P) = G(P), \tag{51}$$

where F and G are functions of P along with flow and material parameters. For granular suspensions, it is not certain that such an equation may still be used. Indeed, several major issues arise due to high concentrations. Typical examples include the development of a layered structure for simple shear flows. Under most flow conditions with solid concentrations in excess of the critical concentration ϕ_c, the particles align themselves into layers oriented in the direction of mean flow, whereas for concentrations below ϕ_c, a random particle microstructure is generally observed. The change in particle arrangement around ϕ_c has been well documented through various experiments and simulations [see for instance the numerical simulations by Campbell and Brennen (1985); Lun and Bent (1994); experiments by Azanza (1998) on the collisional regime; and Gondret and Petit (1995) on the hydrodynamic regime]. Such a transition is reflected at the macroscopic scale by a drop off of viscosity and it seems to be rather general for multibody systems since it may be also observed for colloidal dispersions [Ackerson (1994)] or liquids [Brandt and Bugliarello (1966)]. The layered structure has a significant effect on the flow behavior, which must adjust and in turn modifies the particle organization. Thus it may be expected that a feedback process be required to describe the interplay between micro- and macroscales. Another problem due to high concentrations is the development of strong correlations between neighboring particles and it is not certain that using the replacement of the condition averaged function $\hat{f}^{(2)}$ by f is still meaningful. To our knowledge, a rigorous analytical treatment of the microstructure formation at high shear rates has not been yet provided for concentrated suspensions.

In the following we shall summarize the various approaches attempted in the literature to get round this problem. For each regime, we present the usual assumptions made on the pair distribution function and the associated contribution. As far as we know, to date, no entirely microstructural model has examined transitional regimes. For transitional regimes, the particulate contribution to bulk stress is generally written as a simple addition of the two elementary contributions. Alternative simple models, perhaps more physically based, consist of balancing each elementary contribution depending on the value of appropriate dimensionless numbers. Examples are given in a companion paper [Coussot and Ancey (1999a)]. In the remainder of this paper, we focus our attention on pure regimes.

B. Collisional contribution

Without loss of generality, we retain Newton's law as the collisional law [see Eq. (1)]. In this case, the impulse exchanged during a binary collision may be written

$$\mathbf{J} = \tfrac{1}{2}(1+e)m(\mathbf{k}\cdot\mathbf{c})\mathbf{k}, \tag{52}$$

where $\mathbf{k}$ is the unit vector joining the centers of two neighboring particles and $\mathbf{c}$ denotes the relative velocity of the particle centers. The collisional stress may be defined as the average of the momentum exchange during a collision ($\mathbf{J}$) multiplied by the collision frequency. The next step is to make the Enskog assumption, usual in kinetic theories, which specifies that the velocities and relative position of particles are uncorrelated so that the pair distribution function may be written as the product of the single particle velocity distribution function and the configurational pair-correlation function $p(\mathbf{r})$:

$$P_2(\mathbf{r}_1,\mathbf{c}_1;\mathbf{r}_2,\mathbf{c}_2) = p(\mathbf{r}_1,r_2)P_1(\mathbf{r}_1,\mathbf{c}_1)P_1(\mathbf{r}_2,\mathbf{c}_2). \tag{53}$$

The configuration space $\mathcal{C}^N$ includes the instantaneous velocities and positions of particles. To supplement this approximation, the single particle velocity distribution function is assumed to be Maxwellian:

$$P_1(\mathbf{r},\mathbf{c}) = \frac{n_d}{(2\pi T)^{3/2}}\exp-\frac{(\mathbf{c}-\langle\mathbf{c}\rangle)^2}{T}, \tag{54}$$

where $\langle\mathbf{c}\rangle$ is the mean particle velocity and T the granular temperature. In most theories, even for very high concentrations, the authors have ignored anisotropy in the distribution of collisional contacts and assumed that the $p(\mathbf{r})$ may be approximated by empirical expressions, such as the Carnahan and Starling equation [Savage and Jeffrey (1981)]. Apart from the contribution by Campbell (1986) for two-dimensional shear flows of disks, there is little work performed on the formation of a layered microstructure in the collisional regime and its effect on the function $p(\mathbf{r})$. Using a heuristic model, built to fit its numerical simulations, Campbell (1986) proposed a sophisticated but nonanalytical function $p(\mathbf{r})$ in the form of a sum of Gaussians: $p(2R\mathbf{k}) = p(R\dot{\gamma}/\sqrt{T},\mathbf{k},R,\phi)$.

The collision rate may be evaluated as the probable number of particles in a volume $dV = (2R)^2(\mathbf{c}\,dt.\mathbf{k})d\mathbf{k}$ (with velocities and relative positions within the ranges $d\mathbf{c}_1$, $d\mathbf{c}_2$, $d\mathbf{k}$, where $d\mathbf{k}$ denotes the solid angle centered at $\mathbf{k}$): $P_2(\mathbf{r}_1,\mathbf{c}_1;\mathbf{r}_2,\mathbf{c}_2)\times(2R)^2(\mathbf{c}\cdot\mathbf{k})d\mathbf{k}\,d\mathbf{c}_1\,d\mathbf{r}_2\,d\mathbf{c}_2$. The mean collisional contribution may thus be written

$$\Sigma^{(p)}_{\text{col.}} = n_d\langle(\mathbf{c}\cdot\mathbf{k})\mathbf{J}\otimes\mathbf{k}\rangle = \int\int\int_{\mathbf{c}\cdot\mathbf{k}>0}(\mathbf{c}\cdot\mathbf{k})\mathbf{J}\otimes\mathbf{k}P_2(\mathbf{r}_1,\mathbf{c}_1;\mathbf{r}_1+2R\mathbf{k},\mathbf{c}_2)d\mathbf{k}\,d\mathbf{c}_1\,d\mathbf{c}_2. \tag{55}$$

An analytical expression of this expression depends on the choice of the configurational pair-correlation function $p(r)$. In practice, for lack of an appropriate theoretical model, a Bagnold-type theory is most often used to evaluate the collisional contribution.

C. Frictional contribution

If the bulk behavior in the frictional regime is well known (within soil mechanics), its explanation at the microscopic level is still lacking. The main difficulty in using Coulomb's law (Sec. II B) lies in the determination of the type of contact (slipping or sticking) and the force distribution within the bulk. Experiments and numerical simulations demonstrate that forces are transmitted through a network of particles in contact and shear induced strong (geometrical) anisotropy in the distribution of contacts, leading in turn to mechanical anisotropy of contact forces. The main idea recently developed in the micromechanics of granular media is to link both the probability of contact (texture) and the probability of forces (stress transmission). In this context, the pair distribution function is used as the probability of finding a contact within the solid angle $d\mathbf{k}$, carrying a force $\mathbf{F} = F_n\mathbf{k}+F_t\mathbf{j}$ lying within the range $d\mathbf{F}\mathbf{j}$ a unit vector orthogonal to $\mathbf{k}$, F_n and F_t are, respectively, the normal and tangential components of the frictional force. Using the local Coulomb's law, Eq. (3), the frictional contribution may be written as

$$\Sigma^{(p)}_{\text{frict.}} = Rn_d\langle\mathbf{F}\otimes\mathbf{k}\rangle = Rn_d\int\int P_2(\mathbf{k},\mathbf{F})F_n(\mathbf{k}+x\mathbf{j})\otimes\mathbf{k}\,d\mathbf{F}\,d\mathbf{k}, \tag{56}$$

where x is the mobilized friction coefficient (for $|x| \leqslant \lambda$ sticking contact and $|x| = \lambda$ for slipping contacts). Until recently, little has been done to determine the probability distribution P_2. In most cases, the $\mathbf{k}$ dependence is usually expressed through a second order Fourier expansion. The $\mathbf{F}$ dependence is less obvious. Using statistical arguments, Liu and co-workers argued that the distribution probability must decay as $e^{-\beta F}$ for large forces (with β a parameter). Such a trend has been confirmed by numerical simulations, notably the ones performed by Radjai and Wolf (1998) Radjai *et al.* (1998) on various systems. Using a contact dynamics approach, these authors showed that the normal force distribution is bimodal and may be written as

$$P_{F_n} = \begin{cases} A\xi^{-\alpha} & \xi < 1 \\ Ae^{\beta(1-\xi)} & \xi > 1 \end{cases}, \tag{57}$$

where $\xi = N/\bar{N}$, $A = (1-\alpha)\beta/(1+\beta-\alpha)$ a normalization factor, α and β are two exponents, which vary slightly with shear rate (α is an increasing function of the shear rate and β a decreasing function); in a quasistatic flow, it is found that $\alpha \approx 0$, $\beta \approx 1.4$.

Radjai *et al.* also demonstrated that the mobilized friction coefficient x is randomly distributed between $[0, \lambda]$. This result has important consequences, since it allows one to simply relate the distributions of normal and tangential forces on the basis of Eq. (3). To that purpose, let us consider the joint probability of normal and tangential forces $P_{N,S}(N,S)$. Using the Jacobian of the transformation $(N,S) \rightarrow (N,x)$ and the statistical independence of x with respect to N, we easily deduce

$$P_{N,S}(N,S) = N^{-1}P_{N,x}(N,x) = N^{-1}P_N(N)P_x(x) = \frac{P_N(N)}{\lambda N} \tag{58}$$

for $|x| \leqslant \lambda$ and $P_{N,x}(N,x) = 0$ for $|x| > \lambda$. A simple integration of Eq. (58) associated with the substitution $\xi = x^{-1}$ leads to

$$P_s(S) = \frac{1}{\lambda}\int_{1/\lambda}^{\infty} \frac{P_N(\xi S)}{\xi}\, d\xi. \tag{59}$$

Making allowance for Eq. (57), integration of $P_T(S)S$, with respect to S, gives the mean tangential force

$$\frac{\bar{S}}{\bar{N}} = B(\alpha,\beta) = A\left(\frac{1}{2\beta} + \frac{1}{2\beta^2} + \frac{1}{2(1-\alpha)} + \frac{1}{(1-\alpha)(2-\alpha)}\right). \tag{60}$$

In the simplistic case of a simple shear flow, where the distribution of contact is isotropic [namely the probability of finding contact at $d\mathbf{k}$ is $n_c/(4\pi)$ with n_c the mean contact number per unit volume] and the tangential force is on average oriented in such a way that it is opposed to motion, then it is deduced from Eq. (56) that: (i) there are no normal stress differences, (ii) the normal stress is: $\sigma = n_c n_d R\bar{N}/3$, (iii) the shear τ and normal σ stresses are linearly linked

$$\tau = B\sigma. \tag{61}$$

This relation is known in soil mechanics as Coulomb's law and B is generally written in the form $B = \tan\varphi$, where φ is called the internal friction angle. The internal friction angle does not depend on the local friction coefficient but only on the distribution coefficients α and β.

Naturally, the approximation of isotropic distribution is far from being acceptable since, even at equilibrium (in a gravity field), the distribution of contacts is anisotropic. Moreover, for simple shear flows, experiments and numerical simulations show that shear induces a strong anisotropy of contact as a result of the loss and gain of contacts in privileged directions of deformation [Radjai *et al.* (1998); Rothenburg *et al.* (1989)]. Little is known about this shear-induced process from an analytical point of view. However, it is certain that the normal stress differences are no longer equal to zero. Concerning the link between normal and shear stress, it may be expected that they are still linearly linked (due to the linearity of $\bar{T}$ and $\bar{N}$), even though the linearity coefficient cannot be currently computed due to the poor knowledge of the form of the contact distribution and the dependence of the force distribution coefficients on the shear rate. In practice, using the phenomenological Coulomb law (borrowed from soil mechanics) is a way of evaluating the frictional contribution until a more completed microstructural theory on friction is developed (see the Appendix).

Another striking property of granular assemblies revealed by numerical simulations of Radjai and Wolf (1998) concerns the existence of two populations of particles depending on the force that they transmit [as displayed by the probability density in Eq. (57)]: a ‘‘load-bearing’’ percolating network of contacts carries forces larger than the average force and supports the whole deviatoric stress without inducing energy dissipation. A second network is composed of particles carrying forces smaller than the average force, contributes mainly to the average pressure and is responsible for energy dissipation in the bulk. As pointed out by Radjai and Wolf (1998), a correct description of bulk friction should separate the contribution to bulk stress of each network.

D. Lubrication contribution

If the microstructure of dilute and concentrated suspensions has been extensively studied with fairly good theoretical results, the case of highly concentrated (granular) suspension lags behind and is still debated. There is currently no unanimity about the predominant action (squeezing or shearing motion), or the average particle arrangement, nor the appropriate method for calculating bulk stress. For instance, the bulk viscosity in a simple shear flow may be computed using either an energy balance principle [Eq. (46)] or the bulk stress definition [Eq. (39)], but these two methods provide a somewhat different estimate: although they used the same approximations and hypotheses as Frankel and Acrivos (1967), Goddard (1977) or Van der Brule and Jongshaap (1991) [both using an expression similar to Eq. (39)] they obtained a bulk viscosity about three times lower than the one determined by Frankel and Acrivos. [We point out that a mistake crept into their (Brule and Jongshaap) calculation: the number density $n = 1/(2R)^3$ should be replaced by $n = \phi/(4\pi R^3/3)$.] Here, we only present the reasoning usually followed to derive the bulk viscosity. The first step is to specify the pair distribution function implicitly considered in the calculations. This is usually done by considering a given configuration of particles (generally assumed to be cubic) and by assuming that the face-to-face distance between particles ξ is fixed on average and related to the solid fraction as follows:

$$\frac{\xi}{R} = 2\frac{\zeta}{1-\zeta},$$

with (62)

$$\zeta = 1-\sqrt[3]{\frac{\phi}{\phi_m}},$$

where ϕ_m is the maximum random solid concentration. The pair distribution function may thus be written as

$$P_2(\mathbf{k})\big|_{r=\xi} = \sum_{i=1}^{n_c} \delta(\mathbf{k}-\mathbf{k}_i), \tag{63}$$

where δ is the Dirac function, $\mathbf{k}_i$ denotes the directions of the neighboring particle centers in the considered configuration with respect to the test-particle center, and n_c is the number of indirect contacts. The squeezing contribution may be evaluated from Eq. (4):

$$\Sigma_{\text{sq.}}^{(p)} = \frac{3\pi}{2}\frac{R^3}{\xi}\mu n_d \langle \mathbf{c}_n \otimes \mathbf{k}\rangle. \tag{64}$$

The relative velocity is computed as the average velocity imposed by the bulk flow:

$$\mathbf{c} \approx 2R\bar{\mathbf{L}}\mathbf{k} - 2R\langle \boldsymbol{\Omega}\rangle \times \mathbf{k} = 2R(\bar{\mathbf{d}}\mathbf{k} - (\langle \boldsymbol{\Omega}\rangle - \bar{\omega})\times \mathbf{k}), \tag{65}$$

where $\bar{\mathbf{L}}$ denotes the bulk velocity gradient, $\bar{\omega}$ is the curl of $\bar{\mathbf{L}}$, and $\bar{\mathbf{d}}$ has been defined in Eq. (14). It follows that the squeezing velocity can be written

$$\mathbf{c}_n = 2R(\langle\bar{\mathbf{d}}\rangle:\mathbf{k}\otimes\mathbf{k})\mathbf{k}. \tag{66}$$

The contribution due to squeezing motion is directly deduced from Eq. (4):

$$\Sigma^{(p)}_{\text{sq.}} = \frac{9}{4}\frac{R}{\xi}\mu\phi(\langle\bar{\mathbf{d}}\rangle:\mathbf{k}_i\otimes\mathbf{k}_i)\mathbf{k}_i\otimes\mathbf{k}_i\,. \tag{67}$$

It should be noted that the Newtonian character of bulk stress is dictated by the symmetry of the directions $\mathbf{k}_i$ with respect to the principal directions of the strain-rate tensor. Likewise, the contribution due to shear motion can be evaluated from Eq. (5):

$$\Sigma^{(p)}_{\text{sh.}} = \frac{3}{2}\mu\phi\ln\frac{\xi}{R}[\langle\bar{\mathbf{d}}\rangle-(\langle\bar{\mathbf{d}}\rangle:\mathbf{k}_i\otimes\mathbf{k}_i)\mathbf{1}]\mathbf{k}_i\otimes\mathbf{k}_i \tag{68}$$

and the contribution arising from rotation of particles can be written as

$$\Sigma^{(p)}_{\text{rot.}} = \frac{3}{2}\phi\mu\ln\frac{\xi}{R}[\mathbf{k}_i\times(\bar{\omega}-\langle\Omega\rangle)]\otimes\mathbf{k}_i\,. \tag{69}$$

Let us consider a simple shear flow. If we assume that: (i) the particle configuration is cubic, (ii) its privileged axes coincide with the principal axes of the strain-rate tensor, (iii) the predominant action is due to squeezing, then we may deduce that the bulk viscosity varies as: $\bar{\mu} = aR\mu/\xi$, with $a = 9\phi/4$. Goddard (1977) also found a similar expression, but with $a = 3\phi/8$. The viscosity given by Van der Brule and Jongshaap (1991) is identical to the one computed here provided that the correct definition of the density number is employed ($a = 3\pi/8$, otherwise). Using an energy-based method, Frankel and Acrivos (1967) obtained a similar expression with $a = 9/4$. Sengun and Probstein (1989) inferred a more complicated expression from energy considerations but, asymptotically for solid concentrations near the maximum concentration, they found a comparable expression for the bulk viscosity, with: $a \approx 3\pi/4$, close to the value determined by Frankel and Acrivos. On the basis of energy and kinematics considerations, Marrucci and Denn (1985) argued that coefficient a is not constant and must vary as: $a \propto \ln\xi/\xi$ in the worst case. Likewise, Adler and Brenner (1985) put forward that averaging the different configurations, through which the particle arrangement passes, does smooth the singularity $1/\xi$ and consequently the bulk viscosity does not diverge when the solid concentration tends to its maximum.

This rapid review shows discrepancies in the different approaches followed so far. To date, computations of the bulk viscosity based on energy balance come closer to experimental data. They have also been criticized due to their speculative nature. As pointed out by different authors [Marrucci and Denn (1985); Adler and Brenner (1985)], the mean-field approach presented here suffers a great deal from questionable approximations. Among others, it is obvious from Eq. (68) that the resulting bulk stress tensor depends to a large extent on the particle arrangement, the face-to-face distance between particles, and the velocity field. The thorough examination of elastohydrodynamic processes involved in bringing together two isolated spheres has been performed by Davis *et al.* (1986). It shows that the relative velocity varies to a more or less large extent depending on the Stokes number value, which precludes using a simple mean-field approximation of velocity. A correct treatment of the hydrodynamic regime is still to be achieved.

VI. CONCLUSION

In this paper, we have examined the bulk stress tensor of granular suspensions and the main flow regimes. The major assumption underlying our investigation is the possibility of considering granular suspensions as equivalent continuum media and inferring bulk stress from particle behavior. Our attention has been restricted to simple shear flows for technical and physical reasons. We have used the microstructural approach initially proposed by Batchelor for obtaining the bulk stress tensor of dilute suspensions. Here we have taken advantage of the fact that Batchelor's definition of bulk stress is consistent with most definitions used in mechanics. This consistency may be seen as very useful in the perspective of a unique framework for the treatment of fluid–solid mixtures. The present paper emphasizes the similarities between the various methods used in soil mechanics, suspension rheology, and physics of granular flow to determine the constitutive equation.

A serious difficulty has also been underlined: the equality between the true dissipation rate and the bulk dissipation rate is not satisfied for any flow condition (owing to contact dissipation). This nonequality raises questions about the reliability of some energy-based techniques used in the computation of bulk viscosity. We suggest considering the energy dissipation as an additional equation, required to close the motion equation set (as the fluctuating kinetic energy equation in turbulence).

To evaluate the particulate contribution to bulk stress for each flow regime, we need to know the pair distribution function. We reviewed current knowledge on this function, which mainly stems from the results of numerical simulations. On the whole, little is known about its mathematical structure and most of the current microstructural models for granular suspensions rely on heuristic considerations on its form. This is particularly apparent for the hydrodynamic regime, where the arrangement of particles and force distribution are still controversial (in contrast to dilute and moderately concentrated suspensions). Any progress in the rheology of granular suspension depends on a better understanding of the shear-induced auto-organization of particles.

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APPENDIX

In most current models, the averaged frictional contribution [Eq. (56)] is simply replaced by the term given by Coulomb's law used in Soil Mechanics for (plastic) yielding. It is of interest to specify under which conditions such a phenomenological, appropriate for small-deformation flows, can be used in other flow conditions. In its modern formulation (revisited by Sokolovski, Bishop and Bjerrum, then by Roscoe), this law does state the two following points:

(i) There is yielding within the bulk as soon as Mohr's circle of stress (which represents the state of that element in the Mohr–Coulomb plane $\tau-\sigma$) touches the limiting-stress line given by

$$\tau = \pm\sigma' \tan\varphi, \tag{A1}$$

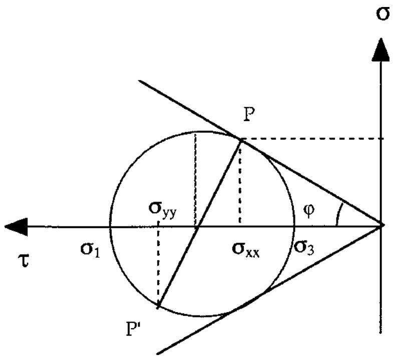

FIG. 3. Mohr's circles for frictional stress and strain (respectively, on left and on right) in the case of a simple shear flow. (Tensile stress is positive.)

where σ' is the effective stress (according to Terzaghi's postulate, it is the total stress minus the interstitial fluid pressure) and φ is the internal friction angle which is a material constant in the critical state theory [Schofield and Wroth (1968)]. This angle must not be confused with the *friction mobilized at the rupture* (which depends on initial conditions and solid concentration). The *critical state* of a soil corresponds to the asymptotic flow condition (reached at large deformations) where a soil element continues to deform without any change in stress and volume. The circle representative of the frictional stress is compelled to move by remaining tangential to the rupture envelope as long as the flow takes place. Before the critical state is reached (namely for any Mohr's circle of stress included in the failure envelope), the deformation is very small and one observes typical hardening elastoplastic behavior; the bulk may be considered as a solid. Using Eq. (A1), we admit that the minor and major principal stresses of the frictional stress are linked by the following relation:

$$\frac{\sigma_1}{\sigma_3} = \frac{1+\sin\varphi}{1-\sin\varphi}. \tag{A2}$$

(ii) The plane along which the rupture develops (with respect to the direction of the major principal stress) should be one of the two planes for which Mohr's circle touches the limiting-stress line (represented by P in Fig. 3). In practice, it is observed that effectively, the most often encountered orientation of the rupture plane is as predicted by Coulomb's law (notably in triaxial tests [Mokni (1992)]). But it is now well established in soil mechanics that the orientation of the rupture planes is not unique [Desrues (1991); Mokni (1992); Vardoulakis and Sulem (1995)].

In our context, we are faced with additional difficulties concerning this formulation appropriate for quasistatic flows. Contrary to soil mechanics, we are looking for the orientation of the rupture surface occurring at large deformations. Moreover, the stress state of the bulk may include several contributions and we can imagine that, in some cases, the shear plane is imposed by processes different from the frictional one. We show in the following that (at least) two orientations are possible depending on the bulk stress state with respect to the Coulombic rupture envelope.

First, in many (and probably most) cases when flow takes place, the bulk stress state is outside the rupture envelope. The rupture plane associated to the plastic yielding coincides with the shear plane. In this case, we have

$$\Sigma^{(p)}_{xy,\text{frict.}} = p\tan\varphi, \tag{A3}$$

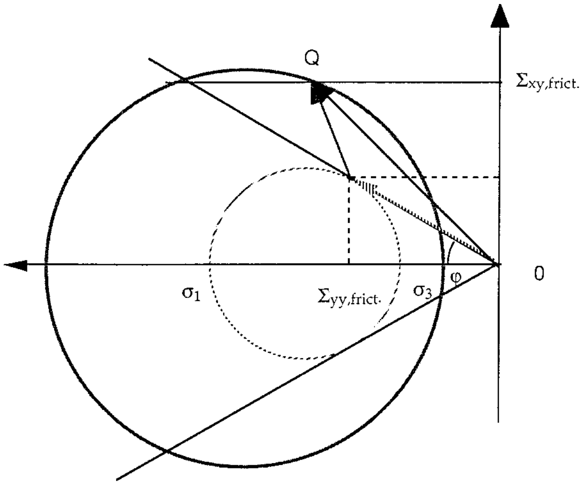

FIG. 4. Mohr's circle for bulk stress and frictional contribution (respectively, continuous and dashed line) in the case of steady frictional-collisional regime (building of the first solution).

with

$$p = \Sigma^{(p)}_{yy,\mathrm{frict.}},$$

where p is the normal component of the frictional contribution. We shall refer to it as *granular pressure*. This case is illustrated by Fig. 4 in the case of a transitional regime (for instance frictional–collisional regime), for which the bulk stresses result from different contributions. The vector **OQ** represents the bulk stress (in the flow plane). This vector may be split into a frictional contribution **OP** and a collisional contribution **PQ**.

Second, in some circumstances, a flow can take place although the bulk stress state is inside the rupture envelope. We shall use an example in Fig. 5 to examine how the rupture surface orientates itself in order to make the flow possible. The total stress may be still split into a frictional contribution (**OP**) and, for instance, a collisional contribution (**PQ**). Under some circumstances, the point Q representative of the bulk stress state is far away from point P. If the point of rupture (P) on Mohr's circle of stress was given by point (ii) of Coulomb's law (namely P in Fig. 5), then the stress vector representative of the collisional contribution can be very large, which may be physically impossible (de-

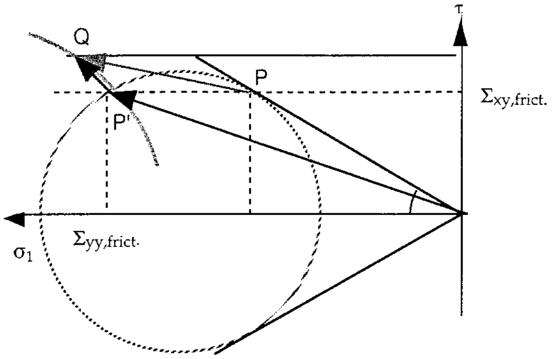

FIG. 5. Building of the second solution.

pending on the expression of the collisional stress). Another way of determining the orientation consists of locating it in P' (see Fig. 5): let us assume that the rupture surface coincides with the plane represented by P'; so, the frictional shear stress is still given by the yielding line (the shear stresses acting on P and P' are equal). It is then straightforward to deduce from Fig. 5 that

$$\Sigma_{xy,\text{frict.}}^{(p)} = \frac{\tan\varphi}{1+2\tan^2\varphi}p. \tag{A4}$$

Since the solution to the yielding problem is not kinematically admissible (from the static point of view) and the orientations given by P and P' are competing, we can expect instabilities to affect the equilibrium.

The next stage should be the tensorial formulation of the frictional contribution. To that end, we refer the reader to the book by Vardoulakis and Sulem (1995), pp. 213–237. Attention should be drawn to the artificial nature a three-dimensional expression of Coulomb's law. A major disadvantage of Coulomb's law is that it leaves the intermediate principal stress indeterminate. This may be very cumbersome since in many examples of simple shear flow (for instance the axysymmetric Couette flow), the intermediate principal stress is included in the deformation plane and hence the frictional contribution cannot be found.

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