Frictional-collisional regime for granular suspension flows down an inclined channel

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Here granular suspensions refer to very concentrated suspensions of particles within a Newtonian fluid. Under certain conditions given in the paper, the bulk stresses mainly result from the combination of frictional and collisional interactions at the particle scale. The corresponding flow regime is called the frictional-collisional regime. The constitutive equation adapted to this regime is not well known. We propose a constitutive model based on the balance between frictional and collisional interactions. We have applied this model to granular flow down an inclined channel. It is shown that the mass flow rate is proportional to the flow depth.

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I. INTRODUCTION

In a previous paper referred to as paper I hereafter [1], we suggested that very concentrated mixtures of noncolloidal solid particles within a fluid should be called ''granular suspensions.'' The introduction of this notion is motivated by the peculiar role played by the solid concentration ϕ (defined as the ratio of solid volume to total volume), as the motion of particles through the bulk is increasingly impeded as the solid concentration increases. When it exceeds a critical value (sometimes called the random loose packing concentration) similar to a dynamical percolation yield (ϕ_c), a continuous network of particles in contact forms throughout the bulk [2]. The formation of this network causes significant changes in the flow behavior: dilatancy, the ordering of particles in layers (for simple shear flows), rearrangement of stress components, the appearance of a minimum in the flow curve, and so on. These abrupt changes around ϕ_c have been shown by several numerical simulations on various systems [3] together with rheometric work [4]. Generally, the authors found that the critical concentration ϕ_c is close to the concentration of a face-centered-cubic arrangement for suspensions made up of identical spherical particles: $\phi_c = \pi/6 \approx 0.52$ ($\phi_c = \pi/4 \approx 0.785$). A second critical value of the solid concentration exists: it corresponds to the random solid concentration ϕ_m, above which it is not possible to add particles without bending them. In the case of monosized sphere mixtures, numerical simulations have shown that this concentration is $\phi_m = 0.635$ [5]. As a consequence, in the range of concentration $[\phi_c, \phi_m]$ particle suspension flows exhibit many peculiarities due to the formation of a particle network, which preclude inferring the bulk behavior by merely extrapolating from dilute or moderately concentrated suspension behavior. Specific theories are needed to model granular suspension flows. Furthermore, owing to the high concentrations, bulk behavior is chiefly dictated by particle-particle interactions (collision, lubricated contacts, friction, colloidal forces, etc.). According to the type of predominant contact, various types of bulk behavior are observed [6].

The constitutive equation of granular suspensions of particles is known only for limiting flow conditions, where only one type of particle contact prevails. Examples include kinetic theories used to infer motion equations for rapidly sheared flows of particles [7]. In this case, we consider that the main interactions between particles are collisions and the part played by the fluid phase in the generation of stress can be disregarded. Likewise, the frictional behavior exhibited at very low shear rates is usually modeled using the phenomenological law of Coulomb [8–10]. In this case, it is shown that the bulk stresses result from sustained contacts between particles, which carry frictional forces throughout the bulk [11]; the role of the fluid phase is mainly limited to the fluid pressure in the pores.

In many cases of interest, the flow regime is intermediate between these two limiting regimes; in other words, both frictional and collisional contacts play a role. For instance, in a geophysical context, gravity-driven flows, such as stony debris flows or rock falls, are presumed to belong to the frictional-collisional regime. In this case, the constitutive equation is poorly delineated. It is now well established that asymptotic theories (such as kinetic theories based on purely collisional interactions) can no longer be used to suitably represent the behavior of frictional-collisional regimes for a wide range of flow conditions [12]. In neighboring scientific areas concerned by the present issue, such as geophysics, scientists are not always convinced themselves of the differences between frictional-collisional and (purely) collisional regimes, and Bagnold-like constitutive models or models adapted from kinetic theories are used.

The development of constitutive equations suitable for describing the frictional-collisional regime has received little attention. As far as we known, the first attempt is due to Savage [13]. In order to fit experimental data obtained on an annular shear cell, he proposed to divide the total shear stress into a part due to a Coulombic frictional contribution (namely, rate-independent part) and a collisional contribution (depending on the square of the shear rate). Further developments were introduced by Johnson and Jackson [14]. Following the suggestion by Savage, Johnson and Jackson expressed bulk stress as the sum of a collisional contribution and a frictional term. More recently, a very similar approach has been used by Jyotsana and Rao [15] to study dry granular confined flows through hoppers. An alternative point of view have been proposed by Mills, Loggia, and Tixier [16]. These authors have modeled dry granular flows as the motion of a network of transient solid chains through an assembly of particles behaving as a viscous fluid. In addition to these constitutive equations examined for particular flow conditions, we can quote the more general tensorial expression obtained by Berker and VanArsdale [10] or the original approach proposed by Savage [17].

From an experimental point of view, little is known about the frictional-collisional regime and, more generally, granular suspension flows. To date, not many experiments have been carried out. Published experimental works have endeavored to measure a few quantities (velocity profile, density profile, etc.) in a narrow range of flow conditions without providing a comprehensive picture of flow pattern (flow regimes, discharge equation, etc.). This perhaps explains why they are not always consistent when compared with each other. For instance, in the case of dry granular flows down inclined channels, some authors found that, for a steady uniform flow, the discharge equation was $q \propto h^{2.5}$ (with h the flow depth and q the flow rate) [18], whereas others found a relation in the form of $q \propto h$ [19]. Another example includes the direction of free surface wave of granular avalanches, which can propagate downwards or upwards depending on the experimental conditions as reported by Douady, Andreotti, and Daerr [20]. Our opinion is that all experimental and theoretical aspects are not necessarily irreconcilable, but, on the contrary, constitute various aspects of a complex flow pattern. To date, the theoretical models (quoted above) fail to describe the observed flow pattern and the paramount features of the frictional-collisional features. In order to gain insight into the behavior of granular suspensions, we suggest studying a granular suspension flow in a simple flow geometry. Here we shall study the case of gravity-driven flows down an inclined, rough, infinite plane. Our model focuses on simple granular suspensions, made up of noncolloidal monodisperse, solid, spherical particles within a Newtonian fluid. First, we outline the definition of the frictional-collisional regime using dimensionless numbers. The second part of the paper is devoted to an overview of the microstructural approach applied to the frictional-collisional regime. In our case, computation of the average stress tensor is limited by poor knowledge of the contact distribution within the bulk. Here we propose a simple model based on a physical analysis of the behavior at the particle level. Emphasis is given to the characteristic times associated with each type of interaction and interplay between them. In a similar way to what was done in the earlier stages of turbulence theory (boundary layer and energy cascade theories), we shall use dimensional arguments and approximate evaluations of physical mechanisms to gain insight into the frictional-collisional regime. As in turbulence or kinetic theories, the constitutive equation must be coupled with the (kinetic) energy balance equation for the motion equation system to be closed. In the third part of this paper, we examine the particular case of gravity-driven flow down an inclined channel. Such a geometry is very appropriate, because the normal and shear stress distribution is perfectly known in a steady uniform flow.

The main distinctive feature in comparison with previous models is the peculiar role ascribed to the kinetic energy balance in the interplay between particle interactions. Indeed, in agreement with most theoretical models developed for the frictional-collisional regime, we find that the shear stress τ is not a simple one-to-one function of the shear rate, but must also depend on the normal stress σ_n and the granular temperature (i.e., the root mean square of the velocity fluctuations) T: $\tau=\tau(\dot{\gamma},\sigma_n,T)$. But whereas most of the available theoretical models have expressed the bulk shear stress as the simple sum of a collisional contribution and a frictional term and have admitted that both elementary contributions are independent, here we explore the possibility of a strong relationship between these two contributions. This relationship is sought via the energy balance equation.

II. TOWARDS A DEFINITION OF THE FRICTIONAL-COLLISIONAL REGIME

In paper I, we showed that, when a granular suspension flow can be regarded as a one-phase flow at the macroscopic level (namely, when there is no significant difference between the mean velocities of each phase), the constitutive equation of the equivalent continuum depends on the type(s) of predominant contact. We define the *frictional-collisional* regime in the following way. It is mainly characterized by the predominance of frictional and collisional interactions between particles within the bulk. Indirect particle interactions (such as lubricated contact) and viscous stresses may be ignored.

It is worth noticing that friction and collision are based on very similar mechanisms at the microscopic level [21]. Their distinction is meaningful only at the particle scale where it is possible to distinguish them by their effects. The treatment of binary contacts is fairly simple. A collision may be defined as a very brief contact whose effect is an exchange of momentum between particles. Thus the contact law is generally sought in the form of a discontinuous change in velocity. Conversely, friction is a sustained contact; the long duration of contact requires a force to be applied to keep the two particles in close contact and accordingly the contact law is expressed as a relation linking the components of the applied force. The treatment of multibody contacts (involved in granular suspension flows) is less straightforward and especially for collisional contacts. Here we suggest defining a collision as a brief exchange of momentum during the impact of two (or more) particles regardless of what happens after the impact (propagation of elastic waves, rebound, frictional contact, sticking). Likewise, we consider friction as a long-range interaction.

On the basis of the above comments, it is possible to propose criteria defining the frictional-collisional regime using dimensionless groups [6,22]. The occurrence of direct contact between particles is conditioned by the collapse (at least in part) of lubricated contacts. In paper I, we suggested employing the Bagnold number defined as the ratio of particle inertia to the work the lubrication force,

$$N_{\mathrm{Ba}}=\frac{\rho_p R^2 \Gamma}{\mu}\,\frac{\delta/R}{\ln \delta/R}, \tag{1}$$

where μ is the fluid viscosity, Γ the mean shear rate, ρ_p the particle density, R the particle radius, and δ the mean particle-to-particle distance. For lubrication effects to be negligible with respect to collisions, the Bagnold number must satisfy $N_{\mathrm{Ba}} \gg 1$. Likewise, in paper I, we have defined the Coulomb number as the ratio of collision magnitude to the typical stress Σ acting on particles:

$$N_{\mathrm{Co}} = \frac{\rho_p R^2 \Gamma^2}{\Sigma}. \tag{2}$$

Introducing dimensionless numbers to characterize bulk behavior of granular suspensions is not new. Many different versions of key dimensionless numbers have already been proposed. For instance, the number that we suggest calling the Bagnold number [Eq. (1)] is formally identical to the Stokes number or the particle Reynolds number, used by other authors.

The frictional-collisional regime is expected to occur when the corresponding contributions in bulk stress have the same order of magnitude, namely, when $N_{\mathrm{Co}} = O(1)$. Naturally, this is a rather crude classification since many parameters controlling dynamics have been omitted. For instance, in a viscous surrounding fluid, collisions between particles involve more complicated mechanisms based on the coupling between hydrodynamics and elasticity as described by Davis *et al.* [23]. In this case, another dimensionless number is required to quantify the capacity of particles to deform due to the action of lubrication forces. But insofar as we focus our attention on chief flow regimes, this classification can provide an approximate and simple way of determining the prevailing particle interactions.

III. CONSTITUTIVE EQUATIONS

A. General expression

In paper I, we showed that the constitutive equation in a frictional-collisional regime may be written as follows. The bulk stress is the sum of a fluid contribution and a particle contribution:

$$\overline{\boldsymbol{\sigma}} = \overline{\boldsymbol{\sigma}}^{(f)} + \overline{\boldsymbol{\sigma}}^{(p)}, \tag{3}$$

where the fluid part may be written

$$\overline{\boldsymbol{\sigma}}^{(f)} = 2\mu \overline{\mathbf{d}} - (1-\phi)\overline{p}_f \mathbf{1} - \rho_f \langle \mathbf{u}' \otimes \mathbf{u}' \rangle, \tag{4}$$

where $\overline{\mathbf{d}}$ denotes the strain-rate tensor, $\overline{p}_f$ is the mean interstitial fluid pressure, ρ_f is the fluid density, and $\mathbf{u}'$ refers to velocity fluctuations. We employ brackets and the over bar symbol to represent ensemble and volume-averaged quantities, respectively. Using the ergodicity assumption, we may replace the volume-averaged terms by ensemble-averaged terms. Moreover, in most cases, the particle contribution outweighs the viscous term in the equation, which can therefore be neglected. This is usually shown by considering a generalized Reynolds number in the form $N_{\mathrm{Re}} = \rho_p R^2 \Gamma/\mu$ or $N_{\mathrm{Re}} = \Sigma/(\mu\Gamma)$. Generally, the large value required for N_{Ba} (for a frictional-collisional regime to occur) implies in turn that N_{Re} must be very large. The particle contribution may be evaluated as

$$\overline{\boldsymbol{\sigma}}^{(p)} = \overline{\boldsymbol{\sigma}}^{(p)}_{\mathrm{col}} + \overline{\boldsymbol{\sigma}}^{(p)}_{\mathrm{frict}} - \phi \overline{p}_f \mathbf{1} - \frac{1}{2} J_p \langle \boldsymbol{\Omega}' \otimes \boldsymbol{\Omega}' \rangle - \rho_p \langle \mathbf{u}' \otimes \mathbf{u}' \rangle, \tag{5}$$

where $\overline{\boldsymbol{\sigma}}^{(p)}_{\mathrm{col}}$ denotes the collisional contribution, $\overline{\boldsymbol{\sigma}}^{(p)}_{\mathrm{frict}}$ the frictional contribution, $\boldsymbol{\Omega}'$ the fluctuations of angular velocity, and J_p the inertia moment. The purely viscous contribution $2\mu\overline{\mathbf{d}}$ in Eq. (4) may be neglected compared with the particulate contributions. It may be shown that the particulate contribution ($\overline{\boldsymbol{\sigma}}^{(p)}_{\mathrm{col}}$ or $\overline{\boldsymbol{\sigma}}^{(p)}_{\mathrm{frict}}$) reflects the effects of local forces at the particle level and may be deduced by averaging the local forces [1,24]:

$$\overline{\boldsymbol{\sigma}}^{(p)} = \frac{R}{V} \sum_{m=1}^{N} \int_{A_p^{(m)}} \mathbf{F} \otimes \mathbf{k}\, d\mathbf{k} = R n_d \langle \mathbf{F} \otimes \mathbf{k} \rangle, \tag{6}$$

where $\mathbf{F}$ is the contact force, $\mathbf{k}$ is the outward normal at the contact point, $d\mathbf{k}$ is the angle around $\mathbf{k}$, and n_d is the number density. In the first term of the equality, we use a volume average of all contact forces acting on the surface $A_p^{(m)}$ of N beads included in a control volume V. The second equality is a simple translation of the first one in terms of ensemble average, which is more usual in kinetic theories or homogenization techniques. In most cases, the ensemble average of a quantity $f(\mathbf{r},t)$ is computed as follows [25]:

$$\langle f(\mathbf{r},t) \rangle = \int_{C^2} P_2(t;\mathbf{x},\mathbf{y}) \hat{f}^{(2)}(\mathbf{x},t;C^N)\, d\mathbf{x}\, d\mathbf{y}, \tag{7}$$

where C^N denotes the configuration of N particles (specified by their positions, linear and angular velocities) in the volume V, and P_2 is the *pair distribution function* defined as the probability that the centers of two spheres simultaneously lie, respectively, in $\mathbf{x}$ and $\mathbf{y}$,

$$P_2(t;\mathbf{x},\mathbf{y}) = \frac{1}{N(N-1)} \int P(t;\mathbf{x},\mathbf{y},C^{(N-2)})\, dC^{(N-2)}, \tag{8}$$

where $C^{(N-2)}$ denotes the remaining $N-2$ particles. Likewise, $\hat{f}^{(2)}$ denotes the conditional averaged function when the position of two spheres is fixed:

$$\hat{f}^{(2)} = \int_{C^2} P(t;N-2|\mathbf{x},\mathbf{y}) f(\mathbf{r},t;C^N)\, dC^{(N-2)}, \tag{9}$$

where the conditional probability $P(N-2|\mathbf{x},\mathbf{y})$ is the distribution probability of the remaining $N-2$ spheres when two spheres are fixed at $\mathbf{x}$ and $\mathbf{y}$: $P(N-2|\mathbf{x},\mathbf{y}) = P(N)/P_2(\mathbf{x},\mathbf{y})$. Most often, it is implicitly assumed that the conditional averaged function $\hat{f}^{(2)}$ may be merely replaced by *f*. For dilute suspensions, apart from systems governed by fluctuations (critical phase transition), such an assumption is generally sound. For concentrated suspensions, due to the development of strong correlations between neighboring particles, it is not certain that replacing the condition averaged function $\hat{f}^{(2)}$ by *f* is still meaningful. Investigations of $\hat{f}^{(2)}$ or P_2 are still scarce and restricted to limiting regimes (frictional or collisional). To our knowledge, no work has been published on this topic for the frictional-collisional regime.

In many cases, in order to close the motion equation set, the energy balance equation is needed. We have shown in paper I that its general expression is

$$\overline{\boldsymbol{\sigma}}:\overline{\mathbf{d}} = \frac{3}{2}\frac{d\rho T}{dt} + \boldsymbol{\nabla}\cdot\mathbf{Q} + \frac{d\overline{\varepsilon}}{dt} - \langle \mathbf{n}\cdot]\sigma\mathbf{u}[\rangle, \tag{10}$$

where $\rho=\overline{\rho}=\overline{\phi}\rho_p+(1-\overline{\phi})\rho_f$ is the mean local density and $\overline{\phi}$ the mean solid concentration. T is the granular temperature ($T=\langle u_i' u_i'\rangle/3$), $\overline{\varepsilon}$ the mean internal energy, $\mathbf{Q}=-\overline{\sigma\mathbf{u}'}$ an energy flux due to thermal motion, and $]\boldsymbol{\sigma}\mathbf{u}[$ denotes the discontinuity of $\boldsymbol{\sigma}\mathbf{u}$ through the particle surfaces oriented by the normal vector $\mathbf{n}$ (due to dissipative contacts), and $\boldsymbol{\sigma}$ denotes the local stress (in the fluid or solid phase). On the right-hand side of Eq. (10), the first term represents the increase in random kinetic energy, the second term stands for the diffusion of energy due to thermal motion, the third term denotes energy loss during inelastic collisions, and the fourth term represents frictional dissipation during slipping contacts. In the absence of frictional (slipping) contact, Eq. (10) is similar to the one found for kinetic theories. It is worth noticing that most authors using a kinetic theory for studying the frictional-collisional regime have continued to employ Eq. (10) without including the frictional dissipation. This omission is not physically sound. Another physical interpretation of Eq. (10) is provided by integrating it over a control volume V. In the case of an isochoric steady flow that we shall study in the next section, we easily find

$$\int_V \overline{\boldsymbol{\sigma}}:\overline{\mathbf{d}}\,d\nu = \int_{\partial V}\left(\frac{3}{2}\rho T\mathbf{u}\cdot\mathbf{n}+\mathbf{Q}\cdot\mathbf{n}\right)dS + \int_V (\dot{\overline{\varepsilon}} - \langle \mathbf{n}\cdot]\sigma\mathbf{u}[\rangle)\,d\nu. \tag{11}$$

The different terms of this equation may be interpreted as follows. The contribution on the left-hand side of the equation represents the energy production rate (supplied to the volume V by shear work). There are two types of energy sink. First, energy may be dissipated by diffusion processes. Two different mechanisms occur. In fact they merely redistribute energy in the bulk without generally contributing to energy decay. The advection of the granular temperature is the balance between the incoming and receding granular temperatures (transported by the mean velocity). It should be noted that in a well-established flow (for instance, down a duct), this contribution is zero. The energy flux $\mathbf{Q}$ transports part of the supplied energy to the boundaries of the control volume, where it may be dissipated. This is the case, for instance, whenever the control volume includes a solid wall. A second type of energy sink includes volume dissipation processes, mainly due to particles. Several elementary processes, such as inelasticity or viscous dissipation within the interstitial fluid, are responsible for the energy decrease. Last, contacts between particles constitute another important dissipation mechanism: since neighboring particles do not move at the same velocity and exert significant forces on each other, mechanical energy is lost and transformed into heat.

The framework presented above is very general and includes most of the available theoretical models dealing with the frictional-collisional regime. In the earlier model proposed by Savage [13] to describe granular flows down inclined rough channels, the collisional contribution was inferred assuming that the pair-distribution function P_2 was the product of Maxwellian single-particle velocity distributions and that particles were smooth, but inelastic with a coefficient of restitution e (kinetic theory). He obtained $\tau_{\rm col}=\tau_{\rm col}(\dot{\gamma},T)$. The frictional contribution was estimated using the empirical Coulomb relationship $\tau_{\rm frict}=p_0\sin\varphi$, where p_0 denoted a mean normal stress and φ the internal friction angle. To close the motion equations, he needed the energy balance equation (10), but he did not take the frictional dissipation $\langle \mathbf{n}\cdot]\boldsymbol{\sigma}\mathbf{u}[\rangle$ into account. To solve the resulting motion equations, he assumed further that the ratio of the mean pressure to the dynamic pressure $[p_0/(\rho_p T)]$ was constant at every depth. His model resulted in linear velocity profiles and a mass flow rate varying as $q\propto h^{2.5}$. He also found that steady flows were possible only within a narrow range of channel inclinations. Compared to the experimental data, such a model is in good agreement (at least qualitatively) with some experimental observations, but fails to describe a large number of experiments. For instance, the prediction of the mass flow rate contrasts with the experimental trend $q\propto h$. Most subsequent models were based on the same approach as the one followed by Savage (coupling a kinetic theory and the Coulomb relationship), but used different assumptions or boundary conditions. But on the whole, predictions of the overall flow features were approximately identical, namely, in partial agreement with all experimental data. This shortcoming may originate in irrelevant approximations or an overly simplified approach, which would have discarded some ingredients. Here, rather than challenge the entire approach followed so far by most authors, we suggest exploring a new direction by examining a different interplay between collisional and frictional contributions while keeping the same ingredients as Savage. Indeed, Savage and subsequent authors implicitly admitted that the two contributions $\overline{\boldsymbol{\sigma}}^{(p)}_{\rm col}$ and $\overline{\boldsymbol{\sigma}}^{(p)}_{\rm frict}$ can be calculated regardless of each other. Here we shall attempt to show that these two contributions may be related and thereby such a coupling leads to very different flow features from the one exhibited by Savage.

B. Proposal of a model

If collision and friction involve the same physical mechanisms at the particle scale (elastoplastic deformations, frictional traction during tangential displacement, creep, etc.), they are associated with very different characteristic times. In a similar way to Mills *et al.* [16] or Vardoulakis and Sulem [9], we consider that, at a given time t, it is possible to distinguish two populations of particles.

Photoelastic experiments and numerical simulations have shown the existence of force networks spanning throughout granular media [11,26–28]. Although the existence of particle networks and different populations is now well established, not many experiments or simulations have been performed to gain insight into the dynamic features of these networks. In Fig. 1, we have reproduced a typical diagram showing the distribution of contact forces within a granular flow down an inclined channel, obtained by numerical simulations using a contact dynamics numerical scheme. Aharonov and Sparks [27] performed numerical simulations, which involved arrays of disks sheared by the motion of the

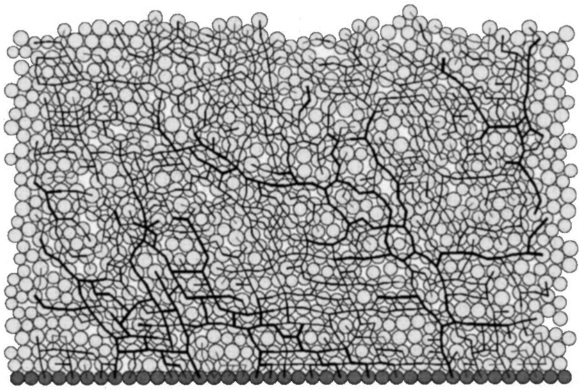

FIG. 1. Normal force diagram in a dry granular flow (Courtesy of F. Chevoir, LMSGC, Champ-sur-Marne, France). The line thickness is proportional to the force strength. The material is made up of polydisperse disks (uniform distribution in size ranging from $0.85R$ to $1.15R$). The channel slope is 18°. The Coulombic friction coefficient is $f=0.4$. The tangential and normal restitution coefficients are zero [45].

upper boundary (Couette flow). They considered two types of boundary conditions. For experiments at constant volume, they observed intermittent networks due to jamming of grains. If the material was free to dilate (free upper boundary with a constant normal force applied), particle networks with two populations of particles were observed. Furthermore, they showed that the density of the sheared material tended towards a constant value (whatever the initial or flow conditions). More recently, Cappart *et al.* [28] performed experiments using an inclined channel with a conveyor belt at the bottom (recirculating system). He employed different kinds of water-saturated mixtures of cylinder-shaped PVC granules. Although they were not able to measure contact forces between grains, they succeeded in measuring the particle velocity and granular temperature at the sidewall. They thus revealed regions where the granular temperature was fairly high and mean velocities were not well correlated and other regions where correlation in the mean particle velocity was significantly enhanced and granular temperature was decreased. Here we shall try to describe some dynamic features of these particle networks using mainly heuristic arguments.

This network of particles in close contact evolves continuously: at any instant, new branches are created, while some links are destroyed. The first category (sometimes called the strong or *competent fraction*) thus includes the particles belonging to these instantaneous networks. If the relaxation time (t_p) for a particle experiencing a force Σ is of the order of the mean life duration of an instantaneous force network (t_n), then the network acts as a rigid, ''load-bearing,'' percolating structure and transmits the gravitational force from upper to lower layers. The characteristic time (t_p) may be evaluated by considering the motion of a particle (of mass m and surface S) undergoing a typical stress Σ: $m\dot{v}=-f\Sigma S$, where f is the friction coefficient. This leads to an estimate of t_p for a spherical particle:

$$t_p \propto \sqrt{\rho_p \frac{R^2}{\Sigma}}. \tag{12}$$

The time t_p is computed as the typical time required for a particle to travel a distance R as a result of the action of Σ (if we assume a zero initial velocity). Computing the time required for an angular displacement of approximately 1 rad, Tkachenko and Putkradze also found that the particle relaxation time is given by Eq. (12) [29]. The life duration of a contact network t_n is of the order of Γ^{-1}, where Γ is the typical shear rate of the flow. Indeed, after a time t_n, two particles which belong to two adjacent layers and are initially in contact must separate. It should be noted that the Coulomb number [Eq. (2)] may also be seen as the square of the ratio of these two characteristic times:

$$N_{\mathrm{Co}}=\left(\frac{t_p}{t_n}\right)^2. \tag{13}$$

Seen as the ratio of a particle relaxation time to a flow characteristic time, the Coulomb number can be interpreted as for the Stokes number (see [30], for instance). For the frictional ($N_{\mathrm{Co}}\ll 1$) or the frictional-collisional [$N_{\mathrm{Co}}=O(1)$] regime, contact is sustained for any particle belonging to the network. Consequently, local dynamical processes are damped and the main interaction between neighboring particles is a Coulombic frictional process. For a network particle, the local motion is then characterized by the relationship between the normal and tangential components of the contact force (respectively, N and S):

$$|S|=\lambda|N|, \tag{14}$$

where λ is the mobilized friction coefficient, whose value depends on the nature of the contact: $\lambda=f$ for a slipping contact and $0<\lambda<f$ for a sticking contact. Equation (14) is known as Coulomb's law or Amontons' law. In the simplistic case of an isotropic contact distribution [namely, the probability of finding contact at $d\mathbf{k}$ is $n_c/(4\pi)$ with n_c the mean contact number per unit volume], it may be inferred from Eqs. (6) and (7) that (i) there are no normal stress differences, (ii) the normal stress is $\sigma_n=n_c n_d R\overline{N}/3$, and (iii) the shear ($\tau$) and normal ($\sigma_n$) stresses are linearly linked [1]:

$$\tau=\eta\sigma_n, \tag{15}$$

where η is a constant. This relation is known in soil mechanics as Coulomb's law, and η is generally written in the form $\eta=\tan\varphi$, where φ is called the internal friction angle. Naturally, in most cases, the contact distribution undergoes a strong shear-induced anisotropy during flow as a result of the loss and gain of contacts in privileged directions of deformation [11,24]. But even in this case, it is expected that the shear and normal bulk stresses are linearly linked (due to the linearity of $\overline{S}$ and $\overline{N}$). The linearity coefficient η cannot currently be computed due to the poor knowledge of the form of the contact distribution and its dependence on the shear rate. Indeed, various approaches have been attempted to determine the pair distribution function P_2: experimental data [31], numerical simulation results [11,26], empirical approximations [32], analogies drawn from the Fokker-Planck equation [33], etc. But they have so far provided only incomplete results. Here, in the absence of a more accurate theory on friction and in accordance with most soil mechanics theories, a practical way of evaluating the frictional contribution due to the competent fraction consists of using the phenomenological Coulomb law (15). We also admit that the parameter φ is intrinsic to the material (it does not depend on the solid fraction) [34]. As the particles belonging to a ''load-bearing'' network carry large forces (larger than the average force), contact is sticking in most cases. This implies that particle dissipate little energy.

A second population (sometimes called the *weak* or *frail fraction*) includes clusters of particles which do not take part in an instantaneous percolating network. As they do not undergo large forces, the typical contact duration is brief and the contact force mainly reflects a momentum exchange,

$$\mathbf{F}=\frac{d(m\mathbf{v})}{dt}, \tag{16}$$

since, over a short time t_c (typically for an elastic collision of two spherical particles $t_c \propto \dot{\gamma}^{-1/5}$), other volume or surface forces may be neglected. If we use the bulk stress definition (6) and assume that the momentum exchanged during a single collision is proportional to $mR\dot{\gamma}$ and the collision rate proportional to $\dot{\gamma}^{-1}$, we may expect the order of magnitude of the collisional contribution to bulk stress to be

$$\overline{\boldsymbol{\sigma}}_{\mathrm{col}}^{(p)} \propto -\rho_p R^2 \dot{\gamma}^2, \tag{17}$$

which is consistent with Bagnold's arguments and predictions of kinetic theories. (As a sign convention, we use positive stress to represent tensile stress.) A more accurate calculation of the collisional contribution requires specifying the pair-distribution function $P_2(\mathbf{r})$ fully and must include the granular temperature. This is achieved by kinetic theories for dilute particle suspensions [7]. For higher concentrations, several major phenomena preclude simply extrapolating the results obtained for dilute suspensions. Typical examples include the development of a layered structure for simple shear flows and the modifications to the contact law. When the particles organize themselves into layers oriented in the direction of mean flow, this causes a strong anisotropy in the pair distribution function, which in turn provokes a significant drop in viscosity [35]. Apart from the contribution by Campbell and Gong [36] for two-dimensional shear flows of disks, little work has been done on the formation of a layered microstructure in the collisional regime and its effect on the bulk stress. When two elastic isolated bodies encounter, the contact is followed by a rebound. For multibody collisions, such a rebound does not necessarily exist. For instance, when throwing a glass bead against an assembly of beads, no rebound is observed. Likewise, when a bead rolls down a bumpy line made up of juxtaposed beads, it can be shown that a collisional process without rebound is the main motion mechanism [37].

Here we shall not attempt to give more details on the form of the collisional contribution and we shall simply assume that the collisional stress components may be written in the following form:

$$\overline{\boldsymbol{\sigma}}_{\mathrm{col}}^{(p)}=\rho_p R^2 \dot{\gamma}^2 \begin{bmatrix} -K_3 & K_2 \\ K_2 & -K_1 \end{bmatrix}, \tag{18}$$

where K_i are dimensionless parameters to be determined. These parameters are necessarily functions of the Coulomb number, since both the contact duration and composition of the bimodal population are functions of this dimensionless number. To justify this assertion, we can propose the following physical scheme. (Such a reasoning is not new; it has been proposed in plasticity [9], thixotropy [6], etc.) Taking into account the bimodal nature of the particle arrangement, we can define a structure state parameter (ζ) as the number of particles in the weak fraction with respect to the total number of particles. The collisional contribution depends on this parameter: $\overline{\boldsymbol{\sigma}}_{\mathrm{col}}=\overline{\boldsymbol{\sigma}}_{\mathrm{col}}(\zeta)$. At leading order, we can estimate that the rate of change of ζ is the difference between the number (per unit time) of network chains destroyed during shear and the number of particles captured by the network. The first term is proportional to a number F (undetermined) of particles available for the weak fraction and to the relaxation time of particles t_p. Likewise, the second term is the product of the network lifetime t_n and the number G of particles which can be included in the network. The balance equation may be written

$$\frac{d\zeta}{dt}=\frac{F(\zeta)}{t_p}-\frac{G(\zeta)}{t_n}. \tag{19}$$

Assuming that F and G are monotonous functions of ζ, we obtain the following relation for a steady state:

$$\frac{F(\zeta)}{G(\zeta)}=\frac{t_p}{t_n} \Rightarrow \zeta=H(N_{\mathrm{Co}}). \tag{20}$$

This demonstrates that the coefficients K must depend on the Coulomb number. Particles belonging to the weak fraction carry forces much lower than the average force transmitted by the network. Contact between particles is most often slipping. Due to inelastic and frictional dissipation, energy loss is significant within the weak fraction.

Finally, we find that for a simple shear flow in a steady state, the bulk shear stress can be written as $\tau=k(\varphi)p+\rho_p R^2 K_2(N_{\mathrm{Co}})\dot{\gamma}^2$, where p still denotes the granular pressure, namely, the mean normal stress carried by the competent fraction. We introduce the friction coefficient k, which is equal to $\tan\varphi$ in most cases [but other values are possible as shown in paper I, where we demonstrated that $k=(1+2\tan^2\varphi)^{-1}$ is possible in some circumstances]. In a manner similar to the fluid pressure for a Newtonian incompressible fluid, we have considered that the granular pressure is *a priori* undetermined. Indeed, owing to the long-range character of friction, the frictional stress state depends on the boundary conditions. Owing to the difference in energy dissipation within each population of particles, the coefficient K_2 must adjust so that shear can dissipate the energy supplied to the system by external forces. In the following we shall investigate a particular class of flow (isochoric steady gravity-driven flow down inclined planes) to examine how this principle of adjustment due to energy balance constraints allows us to infer the variation of K_2 with N_{Co}.

IV. APPLICATION TO FLOW DOWN AN INCLINED PLANE

In this section, we focus our attention on gravity-driven free-surface flows of granular suspension down an inclined

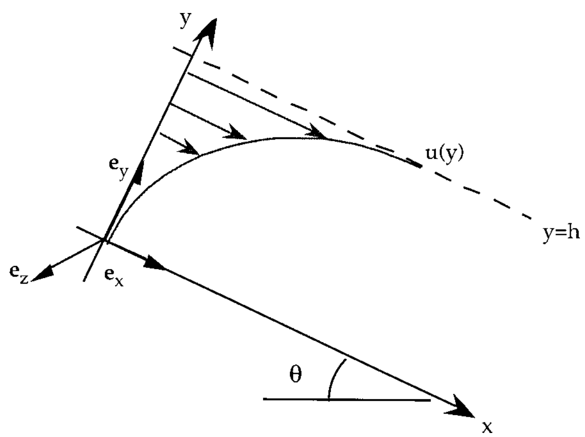

FIG. 2. Definition sketch for steady uniform flow.

plane. It is assumed that (i) a steady uniform regime occurs at an inclination θ to the horizontal, (ii) the bulk undergoes a simple shear, and (iii) the flow is isochoric. The last assumption may be criticized since it is well known that granular flows are dilatant materials. But for dense granular flows, the variations are usually very low (a few percent). Moreover, experiments on channels [38], numerical simulations [27], and arguments stemming from soil mechanics (plasticity theories for large deformations) [9] have shown that the density tends towards a constant value, called the *critical density*.

We use the Cartesian coordinate system of origin 0 and of basis $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ as depicted in Fig. 2. The kinematic field depends on the coordinate y alone and takes the following form:

$$\nu_x = u(y), \quad \nu_y = 0, \quad \nu_z = 0. \tag{21}$$

The strain-rate field is entirely described by the shear rate $\dot{\gamma}$, defined as a function of the coordinate y and implicitly of the inclination θ:

$$\dot{\gamma}(y) = \left(\frac{\partial u}{\partial y}\right)_\theta. \tag{22}$$

Since the variation in density (across the depth) is neglected, we deduce from the momentum balance equation that

$$\tau = \overline{\sigma}_{xy} = g \sin\theta \int_y^h \rho(y)dy \approx \overline{\rho} g \sin\theta (h-y), \tag{23}$$

$$\sigma_n = \overline{\sigma}_{yy} = -g \cos\theta \int_y^h \rho(y)dy \approx -\overline{\rho} g \cos\theta (h-y), \tag{24}$$

where $\overline{\rho} = \phi\rho_p + (1-\phi)\rho_f$ and $\mathbf{g}$, respectively, denote the mean material density and the gravitational acceleration. We need to specify the boundary conditions for stress and velocity fields at the free surface and at the bottom wall. In our particular case, a difficulty arises due to the combination of two coupled interactions. Let us imagine a granular suspension flow down a rough plane with a sufficiently large flow depth. At the free surface, it is expected that the particle contribution to the normal stress σ_n is weak and conversely the particle velocity is large. Accordingly, the local Coulomb number is very large. At the bottom wall, the particle stress is large and the particle velocity close to zero (for a sufficiently rough plane) so that the Coulomb number comes close to zero. Thus we can deduce from these considerations that two boundary layers exist: the first one near the free surface is characterized by the predominance of collision (purely collisional regime), while the second one close to the bottom is governed by friction (frictional regime) as shown in Fig. 3. Due to the complexity of the subject, we have assumed as a first approximation that the flow is deep enough for the thickness of each boundary layer to be neglected. (In the Appendix, we present an approximate treatment of the free surface layer for the particular case of dry granular flow.) Thereby, we assume that there is no slip at the bottom: $u(y) = 0$. Furthermore, we assume that there is no interaction between the free surface and the ambient fluid above (except the fluid pressure). It should be pointed out that for shallow flows, the no-slip assumption no longer holds true. Due to various phenomena (such as depletion, particle size effect, torque transmission, asymmetry of stress tensor, and so on), the flow is influenced by the roughness (see the numerical tests performed by Campbell [39] on boundary interactions). In this case, as pointed out by Brunn *et al.* [40], we can expect a slip velocity in the form $u(0) = (h/R)^\alpha f(\tau_p)$, where α is a parameter tending toward zero when $h \gg R$ and f is a function of the bottom shear stress τ_p. Due to the agitation of particles and the weakness of the normal stress, the Coulomb number is large and the regime is probably collisional. We shall not pursue the matter further here, but in practice the reader must bear in mind that, in the case of a rough plane, for increasing mass flow rate, the regime is probably first collisional, then frictional-collisional; this change must be reflected in the discharge equation.

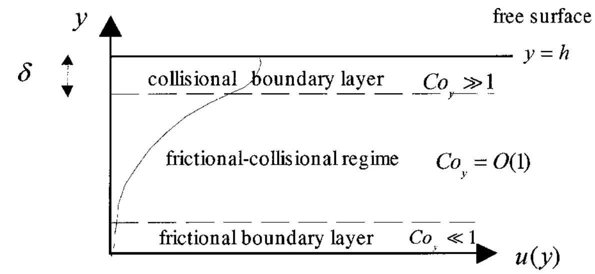

FIG. 3. Sketch for the collisional boundary layer at the free surface.

Now we can write the momentum equations for a granular flow in a steady state [deduced from Eqs. (15) and (18)]:

$$|\sigma_n'| = p + \rho_p R^2 K_1(N_{\mathrm{Co}})\dot{\gamma}^2,$$

$$\tau = k(\varphi)p + \rho_p R^2 K_2(N_{\mathrm{Co}})\dot{\gamma}^2, \tag{25}$$

where σ_n is the normal effective stress (total stress minus fluid stress): $\sigma_n = \rho' g(h-y)\cos\theta$, where $\rho' = \overline{\rho} - \rho_f = \phi(\rho_p - \rho_f)$ is the buoyant density. In Eq. (25) we have expressed coefficients K_i as functions of the Coulomb number only. In the present context of gravity-driven flows down channel, the normal stress and shear rate vary significantly across the flow depth. Accordingly, we use a local Coulomb number N_{Co_y}, whose value is a function of the flow depth. As typical amounts in Eq. (2), we use $\Sigma=\overline{\rho}g(h-y)$ and $\Gamma=\sqrt{g/R}$.

To close the equations, we need to specify the variation of the coefficients K_i with respect to the Coulomb number. We shall use the energy balance equation (10) or its integral form (11) for that purpose. First, in the flow geometry considered here (steady uniform flows), advection of granular temperature is vanishing. In a molecular system, shear work is dissipated into heat, in the form of an increase of the random kinetic energy of molecules. This local heat increase is balanced by a thermal diffusion. In a granular suspension in a frictional-collisional regime, the generation of granular temperature is hindered by several phenomena: proximity of neighboring particles (steric hindrance), nonoverlapping condition due to particle rigidity, and the effect of normal stress, especially for particles belonging to the competent fraction. Thus it may be expected that the granular temperature does not vary in a sufficiently efficient way to be the key parameter of dissipation and its magnitude is approximately $\sqrt{T}\propto R\Gamma/10$. Using Eq. (11), we can evaluate the ratio of the energy dissipated at the channel bottom to the energy supplied by shear. We find $\int_{\partial V}\mathbf{Q}\cdot\mathbf{n}dS/\int_V\overline{\boldsymbol{\sigma}}:\overline{\mathbf{d}}\,dV=O(\Sigma(R\Gamma/10)/(\Sigma h\Gamma))=O(R/(10h))\ll 1$. A third mechanism for energy dissipation ($\dot{\epsilon}$) concerns inelastic loss during collisions, but it is unlikely to play a significant role here. Indeed, if we keep the magnitude of $\dot{\epsilon}$ that we can find using kinetic theories, we can evaluate the ratio of the energy dissipated by inelasticity to the energy supplied by shear: $\dot{\epsilon}/(\overline{\boldsymbol{\sigma}}:\overline{\mathbf{d}})=O(\rho_p(R\Gamma/10)^3/R/(\Sigma\Gamma))=O((R\Gamma)^2/(10^3gh))\ll 1$ for thick flows. Another mechanism is frictional dissipation during slipping contact $[\langle\mathbf{n}\cdot]\boldsymbol{\sigma}\mathbf{u}[\rangle$ in Eq. (10)]. Such a process operates mainly for particles in the weak fraction and can dissipate only a limited amount of energy. Indeed, the energy loss by frictional dissipation is of the order of $\zeta\rho_p(\Gamma+\Omega)Rg$, where $R(\Omega+\Gamma)$ is the relative slipping velocity at the point of contact, for particles belonging to adjacent layers and $\zeta\rho_p\Omega Rg$ for particles of the same layer. For loose systems of particles ($\phi<\phi_c$), the velocity spin generally equals half the shear rate, but for very concentrated systems, there is no general relationship between Ω and $\dot{\gamma}$ due to the absence of correlation between particle spins (frustration) [36].

Thus, from the above arguments, we deduce that Eq. (10) reduces to $\overline{\boldsymbol{\sigma}}:\overline{\mathbf{d}}=\tau\dot{\gamma}\approx-\langle\mathbf{n}\cdot]\boldsymbol{\sigma}\mathbf{u}[\rangle=O(\zeta\rho_p\Omega Rg)+O(\zeta\rho_p\Gamma Rg)$. This leads to the paradoxical result that, on the whole, we should have $(\Sigma-\zeta\rho_pRg)\Gamma\propto\zeta\rho_p\Omega Rg$. That means that energy dissipation is both correlated to the shear rate and uncorrelated. A reasonable assumption can be advanced to overcome this difficulty: for a gravity-driven free surface flow of a granular suspension in a frictional-collisional regime, the mean energy dissipation (per unit volume) is constant at every depth, whose value (Π) depends only on external forces applied to the flow (gravity, in this case). In other words, we may write energy dissipation as

$$\overline{\boldsymbol{\sigma}}:\overline{\mathbf{d}}=\tau\dot{\gamma}=\Pi(\theta). \tag{26}$$

This expression may be also interpreted as follows. The strong fraction enduring frictional sustained contacts is the key ingredient in stress generation. The weak fraction mainly dissipates energy through collisional contacts. Equation (26) means that the shear rate must be adjusted to obtain an equilibrium between stress generation and energy dissipation. As the shear stress is given by Eq. (23), Eq. (26) allows us to deduce the shear rate:

$$\dot{\gamma}=\frac{\Pi(\theta)}{\tau}=\frac{\Pi(\theta)}{\overline{\rho}g(h-y)\sin\theta}. \tag{27}$$

At the free surface, the shear rate should tend towards infinity. In fact, at the same time, the Coulomb number is much larger than unity in this zone and the flow regime must be collisional. A collisional boundary layer must be considered to properly treat the boundary condition at the free surface (see the Appendix). For thick enough flows, the collisional layer at the free surface may be neglected. To be consistent, Eq. (27) must match the expression deduced from momentum equations by eliminating the granular pressure:

$$\dot{\gamma}=\sqrt{\frac{\tau-k|\sigma'|}{\rho_pR^2(K_2-kK_1)}}. \tag{28}$$

A simple comparison of Eqs. (27) and (28) leads to

$$\Pi(\theta)=A\overline{\rho}g\sin\theta\sqrt{Rg\cos\theta(\tan\theta-k\overline{\rho'}/\overline{\rho})},$$

$$(K_2-kK_1)=B\left(\frac{\rho_p}{\overline{\rho}}\right)^2\frac{1}{N^3_{Co_y}}, \tag{29}$$

where A and B are two constants. These expressions hold only for the frictional-collisional regime: $N_{Co_y}=O(1)$. For very large Coulomb numbers, the coefficients K_i tend toward the expression found in the Bagnold theory or kinetic theories. For vanishing Coulomb numbers, they tend toward zero. The velocity profile is deduced by integration of Eq. (28):

$$u(y)=-A\sqrt{Rg\cos\theta\tan\theta-k\overline{\rho'}/\overline{\rho})}\ln\left(1-\frac{\gamma}{h}\right). \tag{30}$$

This expression holds everywhere except near the free surface (see the Appendix). It is worth noticing that the velocity profile is self-similar and convex in contrast with predictions of other models (giving a linear or a concave profile). This is in agreement with experimental data published by Savage [41] on dry granular flow or Lanzoni and Tubino on water-saturated granular flows [42]. The discharge is found to be a linear function of the flow depth:

$$q=Ah\sqrt{Rg\cos\theta(\tan\theta-k\overline{\rho'}/\overline{\rho})}. \tag{31}$$

This result has many practical consequences. Among others, it entails that the mean velocity is independent of the mass flow rate. This is in agreement with experimental data obtained on rough bottom channel [19,41,43]. Furthermore, we can assume that in a way similar to K_2-kK_1, the coefficient K_1 satisfies $K_1=C(\rho_p/\overline{\rho})^2/N^3_{Co_y}$ (where C is a constant) since the simplest way to have $K_2-kK_1\propto N^{-3}_{Co_y}$ is that K_2 and K_1 vary as $N^{-3}_{Co_y}$. Then we find that the granular pressure is a linear function of the flow depth:

$$p = g(h-y)\cos\theta(\overline{\rho'} - A^2C(\overline{\rho}\tan\theta - k\overline{\rho'})). \tag{32}$$

An examination of Eqs. (32) and (30) reveals that a steady uniform flow takes place provided the slope ranges between two critical angles:

$$\tan\theta_2 = \frac{\overline{\rho'}}{\overline{\rho}}\left(\frac{1}{A^2C} + k\right) \geqslant \tan\theta \geqslant \tan\theta_1 = \frac{\overline{\rho'}}{\overline{\rho}}\tan\varphi. \tag{33}$$

For a steady uniform flow to occur, the slope must be in excess of a critical slope (θ_1) so that the shear stress outweighs the Coulomb yield stress. When the slope is increased, the increase in shear rate implies a decrease in granular pressure and eventually, for slopes in excess of a second critical angle θ_2, the granular pressure vanishes; the flow regime is thus collisional again. Such flow partitioning is in agreement with experimental observations [19,22]. Slopes below θ_1 correspond to *immature sliding flows* and slopes in excess of θ_2 correspond to *splashing flows*. The numerical value found for θ_1 is in agreement with our experimental data (dry granular flows) [19] and the one obtained by Tubino and Lanzoni (water-saturated granular flows) [42].

V. CONCLUDING REMARKS

In this paper, we have presented a frictional-collisional model. In the same way as previous models developed for that purpose, the bulk stress tensor is divided into frictional and collisional contributions. This combination is not a simple addition since the two contributions are strongly related via the kinetic energy balance equation. Stress generation is marked by profoundly nonlocal processes since both friction and collision are associated with length correlations over several particle diameters. For friction, we describe this nonlocal character in the same way as for pressure in incompressible fluids by introducing a pressure term, which must be determined by solving the motion equations. For collisions, we ascribe a significant role to energy dissipation. Their effects are strongly dependent on the local balance between competent and weak fractions. As for the thickness of the viscous boundary layer in a turbulent flow, we have considered that the collisional contribution only depends on a dimensionless number (the collisional number). Their variations are governed by the kinetic energy balance. Contrary to simple fluids, several mechanisms are involved in energy dissipation. Due to high concentrations, the classical mechanism of transformation from mechanical energy into heat (thermal motion) probably has limited effects in the energy dissipation. Here we have considered an extreme approximation: the assumption of a constant energy dissipation rate (per unit volume). This corresponds to the case of gravity-driven flow down an inclined channel.

For shallow granular flows [namely, for $h/R = O(1)$], the normal stress due to the particle weight is low and accordingly it is expected that the regime is collisional. For thick enough flows ($h/R > 20$), the collisional regime transforms into a frictional-collisional one. In this paper, this is justified by considering the dimensionless Coulomb number: for the collisional regime, N_{Co} decreases as $H^{-1/2}$ and thus the frictional-collisional is achieved for large flow depth. Anderson and Jackson [12] also found a significant change in the discharge curve ascribed to the transition from a collisional regime to a frictional-collisional regime. The main finding of our model concerns the linearity of the relation between flow rate and flow depth. This point and others are in agreement with experimental data published in the literature.

The present theory is a very crude mean-field approximation, which tries to capture the expected features of particle networks in granular flows and the chief mechanisms of energy dissipation. Improvements or counterarguments should be raised by experiments and numerical simulations in the coming months. Notably, such tests should pay attention to the dynamic characteristics of populations (typical times, evolution, dissipation rate in each population). Furthermore, numerical simulations must be able to specify the pair distribution functions for each population and provide clues about the relationship between these functions and the flow features. Last, the role of the granular temperature both in stress generation and energy dissipation should be better specified. An interesting problem is granular temperature diffusion within clusters of the weak fraction and its influence on the strong fraction.

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APPENDIX

At the free surface, the frictional-collisional regime transforms into a collisional regime. Here, in a similar way to Johnson and Jackson [14], we assume that a collisional boundary layer is superposed on the remaining flow (in a frictional-collisional regime). This boundary layer is characterized by the rapid decrease in the solid concentration. We want to find its thickness (δ) for a given mass flow rate (q). Within this collisional boundary layer, the constitutive equation is given by the following generic expression:

$$\Sigma_{\mathrm{col}}^{(p)} = -(p + \mu_v \nabla\cdot\mathbf{u})\mathbf{1} + 2\mu\overline{\mathbf{d}}, \tag{A1}$$

where p, μ_v, and μ are, respectively, the pressure, the bulk viscosity, and the effective shear viscosity. These parameters depend on the granular temperature of particles (T), the solid concentration (ϕ), the coefficient of restitution (e), and the particle density (ρ_p) and its diameter ($D = 2R$):

$$p = \rho_p f_1(\phi,e)T, \quad \mu = \rho_p D f_2(\phi,e)\sqrt{T},$$

$$Q = -\rho_p D f_3(\phi,e)\sqrt{T}\nabla T + \rho_p D f_4(\phi,e)T^{3/2}\nabla\phi,$$

$$\dot{\varepsilon} = \frac{\rho_p}{D} f_5(\phi,e)T^{3/2}, \tag{A2}$$

where Q denotes the thermal energy flux and $\dot{\epsilon}$ the energy sink. As usual, we shall assume that $f_4=0$. We assume that, due to the rapid decrease in the solid concentration, the mean solid concentration is low in the boundary layer. Accordingly, the kinetic contribution outweighs the collisional part in the constitutive equation. For low solid concentrations, the kinetic model developed by Lun *et al.* [44] is suitable. Keeping only the kinetic part, we can express the functions used in Eq. (2) as

$$f_1(\phi,e)=\phi, \tag{A3}$$

$$f_2(\phi,e)=\frac{\sqrt{\pi}}{14-6e}\left(\frac{1}{g_0(\phi)}+\frac{\phi}{4}(1+e)(3e-1)\right), \tag{A4}$$

$$f_3(\phi,e)=\frac{2\sqrt{\pi}}{(19-15e)(1+e)}\times\left[\frac{1}{g_0(\phi)}+\frac{3}{4}\phi(1+e)^2\left(2e+\frac{1}{2}\right)\right], \tag{A5}$$

$$f_4(\phi,e)=\frac{3\sqrt{\pi}e(e-1)}{2(19-15e)}\frac{1}{\phi g_0(\phi)}\frac{d}{dy}[\phi^2 g_0(\phi)], \tag{A6}$$

$$f_5(\phi,e)=\frac{4(1-e^2)}{\sqrt{\pi}}g_0\phi^2. \tag{A7}$$

Using Eqs. (10), (23), and (24), we directly deduce

$$\frac{d\phi}{dy}=\phi\left(\frac{g\cos\theta}{T}-\frac{T'}{T}\right), \tag{A8}$$

$$\dot{\gamma}=\frac{f_1}{f_2}\frac{\tan\theta}{D}\sqrt{T}, \tag{A9}$$

$$-\frac{\tan^2\theta}{D^2}\frac{f_1^2}{f_2}\phi^2T^{3/2}+\frac{d}{dy}\left[-f_3\sqrt{T}\frac{dT}{dy}+f_4\phi\sqrt{T}\left(g-\frac{dT}{dy}\right)\right]+\frac{f_5}{D^2}T^{3/2}=0. \tag{A10}$$

At the free surface, the boundary conditions are $T'(h)=0$ and $\mathbf{\Sigma}\cdot\mathbf{n}=\mathbf{0}$ with $\mathbf{n}=\mathbf{e}_y$ the normal to the free surface. At the interface with the frictional-collisional zone, we have $u(y_0)=u_{FC}(y_0)$, $\phi(y_0)=\bar{\phi}$. A complete and clearly validated formulation of boundary conditions (at a solid wall) is still lacking: complicated and coupled phenomena (such as torque transmission, depletion, propagation of elastic waves through the bumpy bottom surface) certainly affect the energy balance, but the question of how they interact is quite confused. In most available theoretical treatments, the energy balance is deduced from heuristic considerations and thus involves a series of empirical (indeterminate) parameters. Here we simply assume that the energy balance given by Eq. (10) still holds true, but in accordance with studies on the motion of a single particle down a bumpy [37], it is thought that inelastic dissipation acts as the main sink for granular temperature. Therefore, as a first approximation, we neglect the influence of the thermal energy flux (Q) in the energy balance equation. Finally, we obtain $\tau_0\dot{\gamma}_0=\dot{\epsilon}_{y=0}$, where the subscript 0 refers to the wall position ($y=0$). The coefficient of restitution (at wall), e_w, implicitly used is normally different from the one used in motion equations. Using Eq. (A2), we finally obtain

$$T(y_0)=T_0=\left(\frac{D}{f_5(\bar{\phi})}\bar{\phi}\sin\theta g\,\delta\dot{\gamma}_{\mathrm{FC}}\right)^{2/3}. \tag{A11}$$

As the solid concentration is low, let us introduce a small parameter $\varepsilon=\delta/h$ (much smaller than 1) and let us express the solid concentration and granular temperature as

$$\phi=s_0+\varepsilon s_1+\varepsilon^2 s_2+o(\varepsilon^2), \tag{A12}$$

$$T=t_0+\varepsilon t_1+\varepsilon^2 t_2+o(\varepsilon^2), \tag{A13}$$

where s_i and t_i are functions of y to be determined. Then, using Eqs. (A12) and (A13) in Eqs. (A8)–(A10), we obtain a system of differential equations with powers of ε as parameters. Collecting terms of the same order produces a sequence of equations. For order 0, we have

$$\frac{d^2t_0^{3/2}}{dy^2}=0,\quad s_0'=s_0\frac{g\cos\theta}{t_0}-\frac{t_0'}{t_0}, \tag{A14}$$

and making allowance for boundary conditions, we obtain

$$t_0=T_0,\quad s_0=\bar{\phi}e^{-g\cos\theta(y-y_0)/T_0}. \tag{A15}$$

To order 1, one obtains

$$\frac{8}{5e(1-e^2)}\left[-\frac{5}{2}+\frac{3}{2}(1+e)^2\left(2e+\frac{1}{2}\right)\right]t_1'+gs_1=0,$$

$$s_1'=\frac{s_1g}{t_0}-s_0\frac{t_1'}{t_0}-g\cos\theta s_0\frac{t_1}{t_0^2}, \tag{A16}$$

with no trivial solutions. In order to obtain analytical results, we limit the expansion to terms of order 0. Due to the exponential decrease in the solid concentration, the approximate solution does not provide the position of the free surface. To get around this difficulty, we suggest defining the boundary layer thickness as

$$\delta=\frac{1}{\bar{\phi}}\int_{y_0}^{\infty}\phi(y)dy=\frac{T_0}{g\cos\theta}. \tag{A17}$$

Using Eq. (A9), we deduce the velocity field (to leading order)

$$u(y)=u_{\mathrm{FC}}+\frac{14-6e}{\sqrt{\pi}}\tan\theta\frac{T_0^{3/2}}{g\cos\theta D}(1-e^{-g\cos\theta(y-y_0)/T_0}). \tag{A18}$$

The contribution of the collisional zone to the total discharge may be expressed as

$$q_c=\int_{y_0}^{\infty}\rho_p\phi u(y)dy=\rho_p\bar{\phi}\delta\left(u_{\mathrm{FC}}+\frac{7-3e}{\sqrt{\pi}}\tan\theta\frac{\delta}{D}\sqrt{T_0}\right). \tag{A19}$$

The shear rate is given by Eqs. (28) and (29) and the velocity profile by Eq. (30). At the interface, these expressions may be written as

$$\dot{\gamma}_{\mathrm{FC}}=j(\theta)\frac{1}{\delta} \tag{A20}$$

and

$$u_{\mathrm{FC}}=j(\theta)\ln\frac{h}{\delta}, \tag{A21}$$

where the total depth is $h=y_0+\delta$ and $j(\theta)=\Pi/(\overline{\rho g}\sin\theta)$. The contribution to the total discharge is

$$q_{\mathrm{FC}}=\rho_p\phi\int_0^{y_0}u(y)dy=\rho_p\bar{\phi}j(\theta)h\left(1+\frac{\delta}{h}\ln\frac{\delta}{h}-\frac{\delta}{h}\right). \tag{A22}$$

Using Eqs. (A10) and (A15) together with the definition of the total flow depth and the mass balance leads to a system of four nonlinear equations of variables T_0, y_0, δ, and h. It is worth noticing that the boundary layer thickness is independent of the mass flow rate (as a first approximation):

$$\delta=\frac{(Dj(\theta))^{2/3}}{\sqrt[3]{g}}\frac{(\bar{\phi}\sin\theta/f_5)^{2/3}}{\cos\theta}. \tag{A23}$$

It follows that the influence of the boundary layer is particularly marked for low-mass flow rates (in accordance with experimental observations such as the ones performed by Johnson and Jackson [14]). For sufficiently large discharges, the error caused by ignoring the boundary layer is negligible.

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