Fundamentals of Cosmic Physics

GENERAL INFORMATION

Aims and Scope

Fundamentals of Cosmic Physics is devoted to publishing comprehensive review articles in all the important areas of cosmic physics, including general relativity and gravitation, astronomy and astrophysics, space and planetary physics and geophysics.

The aim of the journal is to publish papers that, while highlighting the current status of research in the field and serving as a guide to the original literature, are internally complete, incorporating the full derivations essential to the topic at hand. The fundamental nature of these papers makes them ideal reference works for graduate students and professionals in the above and related areas.

The editors plan to collect key articles into hardcover books so that a series of Handbooks emerges with several volumes for each area of cosmic physics. Scientists who wish to contribute to *Fundamentals of Cosmic Physics* or who wish to suggest useful topics should write to either editor. Authors should make their articles suitable for classroom presentation to graduate students and to professionals in general.

Notes for contributors can be found at the back of the journal.

Fundamentals of Cosmic Physics, 1991, Vol. 15, pp. 1–106
Reprints available directly from the publisher
Photocopying permitted by license only

Printed in the United Kingdom

Similarity and Self Similarity in Fluid Dynamics

JULIO GRATTON†

PRIFIP-CONICET, LFP, Departamento de Fisica, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria, 1428 Buenos Aires, Argentina

The primary goal of this work is that of providing the readers who are not familiar with the methods of dimensional analysis and similarity the basic ideas and the necessary tools to treat practical problems in fluid dynamics and its applications to problems of cosmic physics.

Contents

1. Introduction 2
2. Similarity and Dimensional Analysis 4
   2.1 Geometrical similarity 4
   2.2 Scaling laws 7
      2.2.1 The theorem of Pithagoras 8
      2.2.2 The formula for the surface of an ellipse as a consequence of affinity 9
   2.3 Physical similarity 10
   2.4 Application: the motion of a body in a fluid 14
   2.5 Two astronomical examples 15
      2.5.1 The gravitational bending of light by the Sun 16
      2.5.2 The advance of the perihelion of Mercury 16
3. The Concept of Self Similarity 19
   3.1 The diffusion of heat 19
   3.2 The laminar boundary layer 22
4. Self Similarity and Intermediate Asymptotics 26
   4.1 Flow past a wedge: self similarity of the second kind 28
   4.2 The modified heat diffusion problem 30
   4.3 Complete and incomplete self similarity 36

†Researcher of the Consejo Nacional de Investigaciones Cientificas y Técnicas, Argentina.

4.4 Self similarity of the first and second kind 42
4.5 Self similarity and groups of transformations 43
5. Self Similar Solutions in Gas Dynamics 45
5.1 The phase plane formalism 46
5.2 Self similar unsteady gas flows 50
5.3 Solutions with discontinuities 51
5.4 Particle trajectories and characteristics 54
5.5 Unsteady planar flows 55
5.5.1 The expansion of a gas into the vacuum 56
5.5.2 The centered compression wave 57
5.6 Strong explosions 60
5.7 Implosions 71
6. Self Similar Shallow Water Gravity Currents 85
7. Self Similar Creeping Flows 89
7.1 The breaking of a dam containing a viscous liquid 93
7.2 Waiting-time solutions 95
7.3 The collapse of a converging viscous current 97
7.4 Progressive waves and limiting to self similar solutions 100
8. Final Remarks 103
Acknowledgements 104
References 104

1 INTRODUCTION

The symmetry of systems, viz., the property of remaining unchanged (invariant) when certain transformations are performed, has important consequences such as the conservation of physical quantities. One can take advantage of this fact to achieve useful simplifications in solving problems. For example, the homogeneity of space implies that an isolated system is translationally invariant, and from this it ensues that momentum is conserved (Landau and Lifschitz, 1959a); this fact allows to simplify the analysis of the motion of a system of interacting particles, because it is posible to investigate the movement of the center of mass independently of the motion of the particles with respect to it. In a similar fashion, an isolated system is rotationally invariant due to the isotropy of space, with the consequence that its angular momentum is conserved; it is well known how this circumstance simplifies the description of planetary motion. Many other examples can be given, that show that the analysis of symmetry properties is a powerful tool for investigating physical phenomena.

The symmetries we have mentioned above are based on geometrical properties, either characteristic or the space-time manifold in which physical phenomena are embedded, and/or specific of the particular

problem one is considering. However, not all the symmetries that appear in Physics are purely geometrical. The reason is that physical quantities are characterized by their dimensions, in addition to their geometrical attributes. The dimension of a quantity is related to the units in terms of which it is measured. Owing to their dimensional properties, the quantities that describe a physical system have symmetries that are related to the fact that the choice of the units of measurements is arbitrary, and bears no relationship with the substance of the phenomena. This is in essence what is called scale symmetry, and its manifestation consists in that the description of physical phenomena must be invariant with respect to changes of the units of measurement, or, equivalently, with respect to the scaling of the quantities themselves.

Dimensional analysis exploits the invariance with respect to the group of scale transformations by reducing the number of combinations among the variables and parameters that govern a problem, and restricting the type of functional dependencies among them, thus simplifying the analysis and allowing to derive useful scaling laws. Sometimes a problem is invariant with respect to a larger group and further restrictions on the number of parameters and their relations are obtained.

Self similarity results when the symmetry of a physical problem leads to a reduction in the number of the independent variables. In this way a considerable simplification is achieved, that frequently allows the analytical treatment of the problem. Very elegant solutions are thus derived. Usually the self similar behavior appears in the intermediate asymptotics of phenomena, when certain details of the initial or boundary conditions are no longer relevant, so that the corresponding parameters can be ignored. The peculiarities of the passage to the limit that leads to the intermediate asymptotics of a given problem allows to classify the similarity solutions as self similarities of the first and second kind. Self similarity of the first kind can be established by dimensional analysis (eventually supplemented by other symmetry considerations). The self similarities of the second kind cannot be derived in this way: it is necessary to follow the evolution of the solution either experimentally or numerically until it passes into its self similar asymptotics, or they can be obtained by direct construction. In the second case this process leads to a nonlinear eigenvalue problem.

This paper is organized as follows:

In Section 2 we introduce in a simple and intuitive way the notion of scale symmetry (similarity) in physics, and its connection with dimensional analysis; some of its consequences are illustrated by means of examples.

In Section 3 we present the concept of self similarity, which is of great importance not only in the Mechanics of Fluids, but also in many other branches of Physics. We show how self similarity permits to simplify problems by reducing the number of independent variables.

In Section 4 we discuss the role of dimensional analysis in establishing the self similarity of a problem; we introduce the self similarities of the second kind, that cannot be established by dimensional analysis alone but require to solve a nonlinear eigenvalue problem to determine the self similar variable, and we discuss the intermediate asymptotic character of the similarity solutions.

In Sections 5, 6 and 7 we develop the phase plane formalism, that allows to find self similar solutions in various problems of Fluid Dynamics, and discuss several applications to the dynamics of gases, the shallow water theory, and to nonlinear diffusion-type phenomena, exemplified by viscous gravity currents.

This work is an outgrowth of the notes of my lectures of 1988 and 1990 on Dimensional Methods and their Applications (a postgraduate Course of the Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires), and on Self Similarity in Fluid Dynamics at the International School on the Physics and Mechanics of Fluids at Tandil, Argentina, in 1989.

2 SIMILARITY AND DIMENSIONAL ANALYSIS

2.1 Geometrical Similarity

The idea of physical similarity is a generalization of geometrical similarity. Accordingly, we shall begin with a brief review of this simpler concept. In its simplest form, we can define geometrical similarity by saying that two figures are similar if all the ratios between corresponding lengths are identical. Thus the polygons in Figure 1 are similar because

$$\frac{l_1'}{l_1} = \frac{l_2'}{l_2} = \cdots = r. \tag{1}$$

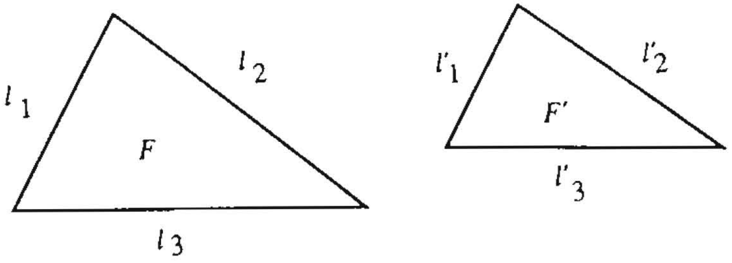

FIGURE 1 Two similar polygons.

The ratio r is called similarity ratio, scale factor, or briefly scale.

A similarity transformation:

$$F \rightarrow F', \tag{2}$$

is effected by means of a change of scale, or scaling:

$$l_1' = rl_1;\ l_2' = rl_2;\ \ldots, \tag{3}$$

i.e. all the lengths l_i' of F' are obtained multiplying the corresponding lengths l_i of F by the scale factor r.

A related but more general concept is that of affine similarity, or affinity. We speak of affinity when there is similarity, but only with reference to a particular system of parameters. Let us consider an example, for more clarity: imagine we have chosen in the plane of Figure 2 a particular system of cartesian axes (x, y). If $P = (x, y)$ represents a point of a certain figure F, and $P' = (x', y')$ represents the corresponding point P' of the figure F', we say that F and F' are affine (or that there is an affine similarity between them) if

$$\frac{x'}{x} = r_x = \text{const}_x,\ \frac{y'}{y} = r_y = \text{const}_y, \tag{4}$$

for any pair of corresponding points of F and F'.

It can be observed in Figure 2 that any two ellipses F, F' are affine, if we refer them to a system of axes whose origin is at the center of the figures, and that is oriented along their principal axes. This choice, with respect to which the affinity is defined, is the particular system of parameters we mentioned above. Recall that a simple method for the construction of ellipses is based just on the affinity between the ellipse and the circle.

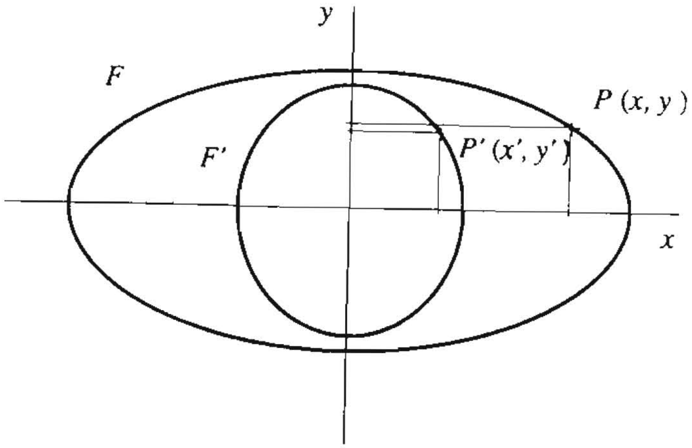

FIGURE 2 Two affine ellipses.

An important concept related to any type of transformation (and in particular to similarity and affine transformations, or mappings) is that of invariant. An invariant is an quantity that is not changed by the mapping (in the present case by the similarity, or the affinity). For example

(a) Consider the angle α whose vertex is at O, and whose sides are OA and OB (Figure 3). Let s denote the arc of a circle whose center is O and whose radius is r, subtended by α. Let us consider a similarity mapping

$$(r, s) \rightarrow (r', s'), \tag{5}$$

that transforms s and r in a new arc s' and a new radius r' (Figure 4).

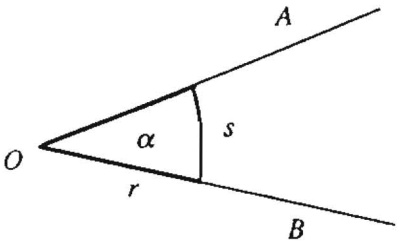

FIGURE 3 Arc of a circle.

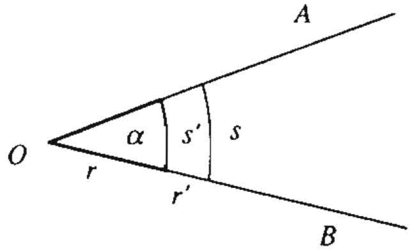

FIGURE 4 The angle is scale invariant.

Clearly the ratio between the arc and the radius (i.e. the angle subtended by the arc) is an invariant:

$$\alpha = \frac{s}{r} = \frac{s'}{r'} = \text{const.} \tag{6}$$

Then angles are scale invariant. On the other hand it is easy to see that angles are not affine invariants.

(b) Consider the ratio between the surface and the linear size of the rectangular elements of Figure 5. Clearly

$$\sigma = \frac{dS}{dx\, dy} = \frac{dS'}{dx'\, dy'}, \tag{7}$$

so that σ is scale invariant, and in this case it is also an affine invariant.

2.2 Scaling Laws

The existence of similarity and affinity invariants allows to derive scaling laws. For instance, if S and S' are the surfaces of two similar figures F and F', and if l and l' denote any two corresponding lengths

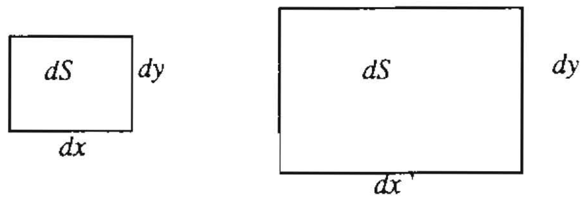

FIGURE 5 The relationship between surface and linear size.

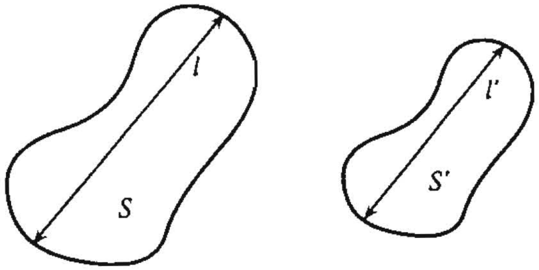

FIGURE 6 The scaling law of the surface.

associated to F and F' (Figure 6), one has:

$$\frac{S}{l^2} = \frac{S'}{l'^2} = \Pi = \text{const.}, \tag{8}$$

and from this we obtain the scaling law:

$$S = \Pi l^2, \tag{9}$$

that expresses that the surface of any geometrical figure is proportional to the square of its linear size. Here Π can only depend on other invariants that determine the shape of the figure (for example, for a polygon, these invariants will be angles and ratios between the sides).

As applications of the surface scaling law we shall demonstrate the theorem of Pithagoras, and derive the formula for the surface of an ellipse.

2.2.1 *The theorem of Pithagoras*

Consider the rectangular triangle of Figure 7, whose sides are a, b, c. A line perpendicular to a, passing through the opposite vertex, divides it into the two triangles denoted by 1 and 2. The surface of the original triangle is equal to the sum of the surfaces of its two parts:

$$S_{abc} = S_1 + S_2. \tag{10}$$

Notice that the three triangles (abc), 1, 2 are similar. Now, for any rectangular triangle one must have

$$S = \Pi h^2, \tag{11}$$

where h denotes the hypothenuse, and the invariant Π can only depend

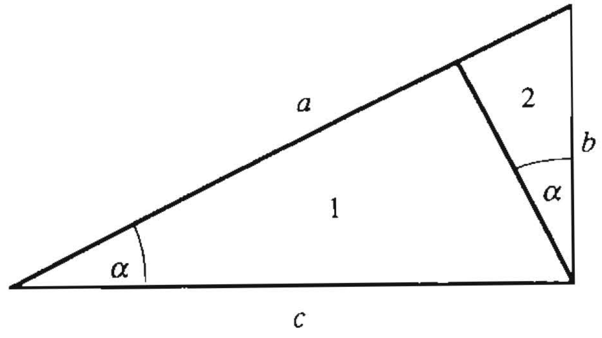

FIGURE 7 The theorem of Pithagoras.

on other invariants that determine the shape of the (rectangular) triangle; then

$$\Pi = f(\alpha), \tag{12}$$

where α denotes one of the angle adjacent to the hypothenuse, since the knowledge of α is sufficient to determine the shape of a rectangular triangle. Using (11) and (12) in (10) on obtains

$$f(\alpha)a^2 = f(\alpha)b^2 + f(\alpha)c^2, \tag{13}$$

and canceling the common factor we obtain the result:

$$a^2 = b^2 + c^2. \tag{14}$$

It is left as an exercise for the reader to explain why the same result cannot be obtained if the triangle is not planar (Migdal, 1977, see also Barenblatt, 1979).

2.2.2 *The formula for the surface of an ellipse as a consequence of affinity*

Figure 8 depicts an ellipse whose half axes are a, b, and whose surface is S_e. We have already shown that the ratio

$$\frac{S_e}{ab} = \Pi_e, \tag{15}$$

is an affine invariant, and in the present case must be a numerical constant, since the ellipse is completely defined by its half axes. Then the

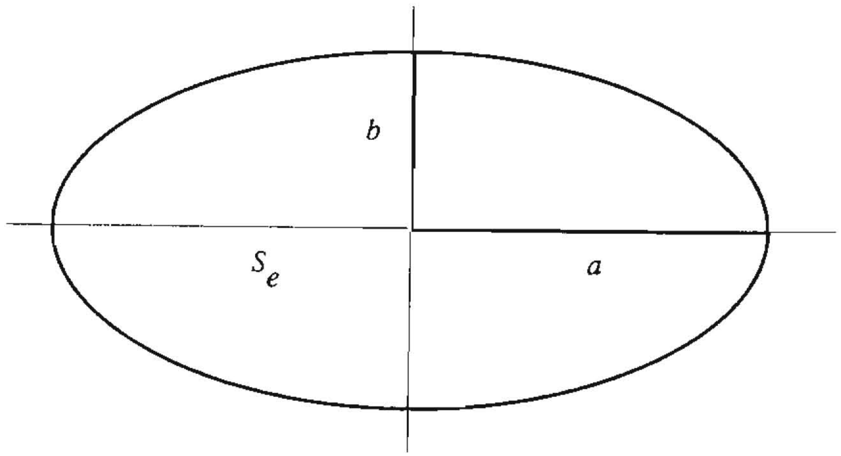

FIGURE 8 The surface of an ellipse.

following scaling law holds:

$$S_e = \Pi_e ab. \tag{16}$$

Since the number Π_e must be the same for any ellipse, it can be determined once for ever using an ellipse of our choice. In particular, the circle is an ellipse with equal half axes. Then

$$\Pi_e = \pi = 3.1415926\ldots. \tag{17}$$

The formula for the surface of an ellipse is then

$$S_e = \pi ab. \tag{18}$$

We now discuss similarity in physics.

2.3 Physical Similarity

Similarity in physics is analogous to geometrical similarity, provided due attention is paid to the fact that physical quantities are characterized by other dimensions in addition to those of geometrical character. We say that two physical phenomena are similar when the characteristics of one of them can be obtained from the characteristics assigned to the other by means of a simple change of scale. Such a change

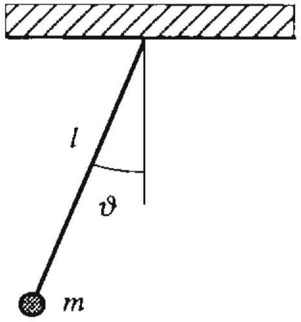

FIGURE 9 The pendulum.

of scale is analogous to the transformation from a system of units of measurement to an other.

To carry out the transformation it is necessary to know the scaling factors. Physical similarity is the basis for employing scaled down models to study the behavior of systems and devices of large size in the laboratory.

Nothing is best than an example to clarify the concepts involved in physical similarity. Consider, say, a pendulum. We shall see that the motion of a particular pendulum belongs to a class of similar phenomena; this is a consequence of the scale invariance of the equation of motion

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta, \tag{19}$$

in which θ denotes the angle of the pendulum with the vertical, l is the length of the string, and g is the acceleration of gravity (see Figure 9). The scale invariance of (19) can be verified explicitly: if all lengths are scaled by a factor R_l and all times by a factor R_t, one obtains (we denote with $'$ the scaled quantities):

$$l' = R_l l;\ t' = R_t t;\ \theta' = \theta;\ g' = R_l R_t^{-2} g, \tag{20}$$

and substituting in the equation of motion:

$$\frac{d^2\theta'}{dt'^2} = -\frac{g'}{l'}\sin\theta', \tag{21}$$

so that the transformed quantities satisfy the same equation as the original ones: the equation of motion does ot change. The consequence of this invariance is that the characteristics of the motion of a pendulum can be obtained from the characteristics of the motion of an other pendulum, by means of a simple change of scale. Notice that the initial conditions (that do not appear in the equation of motion) must also be included in the change of scale.

The scale symmetry can be made evident if the equation of motion is written in terms of the scale invariants of the problem, i.e.

$$\theta\ ;\ \tau \;=\; t/T;\quad \Pi \;=\; T^2 g/l, \tag{22}$$

in which T denotes the period of the oscillation:

$$\frac{d^2\theta}{d\tau^2} \;=\; -\Pi \sin\theta. \tag{23}$$

Since this equation is written entirely in terms of invariants, is itself manifestly invariant.

From the invariant Π one derives the scaling law of the period:

$$T \;=\; (\Pi l/g)^{1/2}. \tag{24}$$

Notice that Π must be a function of the constant invariants of the problem: θ_0, the amplitude of the oscillation, and ϕ_0, the initial phase. But obviously the period cannot depend on the initial phase, so that we can write:

$$\Pi^{1/2} \;=\; f(\theta_0), \tag{25}$$

then one obtains

$$T \;=\; \sqrt{\frac{l}{g}}\, f(\theta_0). \tag{26}$$

In the limiting case of small amplitude oscillations ($\theta_0 \to 0$), Π must be independent of θ_0, consequently f must tend to a constant number (the value of this numerical constant is of course $1/(2\pi)$, but clearly this value cannot be derived by means of dimensional considerations alone).

Based on this example we can make some generalizations, that are consequence of the above mentioned fact that the choice of a system of units is arbitrary, and has no connection with the substance of the phenomenon:

(a) Scale invariants are always dimensionless quantities, that is, quantities whose value is independent of the choice of the system of units. The invariants are constructed combining the dimensional variables, parameters, and physical constants of the problem.

(b) Any physical relation corresponding to a given problem (equations of motion, equilibrium conditions, initial and boundary conditions, etc.) can be expressed as a relationship between scale invariants.

(c) Two phenomena are similar if, and only if, all their dimensionless variables and parameters have the same numerical values.

Dimensional Analysis allows generally (later on we shall show that there are some limitations) to determine the dimensionless combinations appropriate to each particular problem. The Pi Theorem of Buckingham (see for example Bruhat, 1963, Sedov, 1959, Li and Lam, 1964) allows to determine the number of independent dimensionless combinations that can be formed from the quantities corresponding to a given problem: if n is the number of the characteristic dimensional parameters of the problem (constant or variable), and among them there are k that have independent dimensions, the number of independent dimensionless combinations that can be formed is equal to $n - k$.

Scale symmetry and its consequences are always fruitful:

(a) If the governing equations of the prblem are known, the parameters, variables, and constants are determined by inspection, and are the basis for discussing similarity, for the dimensional considerations, and for obtaining scaling laws. In this case scale symmetry usually simplifies the investigation by allowing to reduce the number of parameters, and by imposing restrictions on the type of functional dependencies.

(b) It may be impossible to solve the problem by the processes of analysis and calculations because of overwhelming mathematical difficulties, or because we lack a mathematical formulation of the problem (due to the great complexity of the phenomenon under study, or to insufficient knowledge). In these cases scale symmetry and dimensional arguments are still useful, since they allow to investigate experimentally the problem by means of models of a convenient scale, or because they yield in a simple straightforward way approximate and/or qualitative theoretical answers. Sometimes this may be all that is required, or that we may reasonably hope to obtain. Finally, this type of analysis may throw some light on the nature of the knowledge that is missing in the

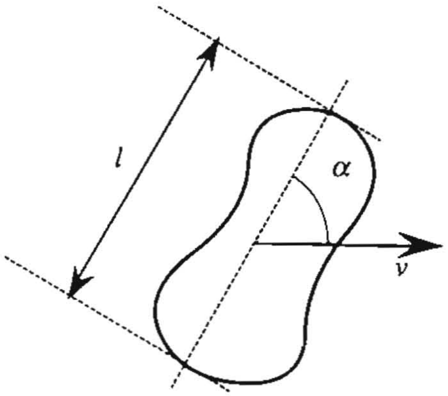

FIGURE 10 A body that is moving in a fluid.

formulation of the problem, thus pointing the direction for further investigation.

2.4 Application: The Motion of a Body in a Fluid

This problem is of paramount importance because of its many applications to engineering and natural sciences. We shall briefly discuss it to show how to take advantage of scale symmetry to simplify the analysis (see Sedov, 1959).

Consider a body that is moving inside an infinite fluid with a uniform translational velocity v, which we shall assume to be much less than the speed of sound in the fluid. Suppose that the shape of the object does not change, so that it is completely specified by giving a characteristic length, l, and a certain number of constant scale invariants that determine its shape. We shall lump together the latter in a form factor f (itself a constant invariant). The motion of the fluid will be completely determined if we fix the angles that give the orientation of the object with respect to v, which we shall denote collectively by α (Figure 10). We shall take into account the inertia of the fluid, determined by its density ρ, and its viscosity, given by the corresponding coefficient μ. For simplicity we shall assume that no body forces (such as gravity) are acting. Then, if we observe the phenomenon from a reference frame moving with the body, the motion of the fluid will be determined by the

following parameters:

$$l,\ v,\ f,\ \alpha,\ \rho,\ \mu. \tag{27}$$

We want to know the force F that the fluid exerts on the body (that is usually called the total drag). Clearly the ratio

$$\frac{F}{\rho l^2 v^2} \tag{28}$$

is an invariant. This invariant must be a function of the other invariants of the problem, namely α, f, and the Reynolds number

$$\mathscr{R} = \frac{\rho v l}{\mu}. \tag{29}$$

Then

$$F = \rho l^2 v^2 g(\alpha, f, \mathscr{R}). \tag{30}$$

The determination of the function $g(\alpha, f, \mathscr{R})$ is a fundamental problem of aerodynamics and hydrodynamics. Let us see what information about it can be derived from dimensional analysis.

Since μ enters in g only through $\mathscr{R}$, some general conclusions concerning the role of viscosity can be derived from the formula (29). First, the effect of viscosity diminishes as $\mathscr{R}$ is increased; if viscosity is ignored ($\mu \to 0$) we arrive at the concept of an ideal fluid. Then in the limit $\mathscr{R} \to \infty$, F must be independent on the viscosity. Therefore, for fast motion ($\mathscr{R} \to \infty$) one must have:

$$F = \rho l^2 v^2 g_1(\alpha, f), \ (\mathscr{R} \to \infty). \tag{31}$$

In the opposite limit (slow motion) viscous forces dominate over the inertia of the fluid. Then in the limit $\mathscr{R} \to 0$, ρ must drop out from the expression of F; consequently in this limit $g \propto 1/\mathscr{R}$. We then obtain:

$$F = \mu v l g_2(\alpha, f), \ (\mathscr{R} \to 0). \tag{32}$$

Then we conclude that Stokes's law is correct for objects of any shape if the inertia terms in the Navier–Stokes equation are neglected.

2.5 Two Astronomical Examples

To conclude this Section we shall apply dimensional analysis to a couple of problems of interest in Astronomy.

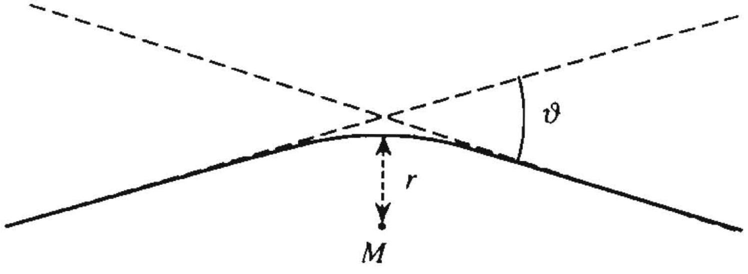

FIGURE 11 Gravitational bending of a ray of light.

2.5.1 *The gravitational bending of light by the Sun*

Let us consider the gravitational bending of light by a point mass M. In Figure 11 we sketch the geometry of the problem. Clearly, the deflection angle θ must depend on M, c (the speed of light), G (the gravitational constant), and r, the closest distance of approach of the ray from M:

$$\theta = f(M, c, G, r). \tag{33}$$

The invariants of this problem are θ and $\phi = GM/c^2r$, so that (33) must be of the form

$$\theta = \Phi(\varphi) \tag{34}$$

To determine the form of Φ, we notice that for $\varphi \to 0$, $\lim_{\varphi\to 0}\Phi = 0$, and that $\lim_{\varphi\to 0}(d\Phi/d\varphi)$ exists. From this it follows that in this limit one must have

$$\theta \cong K\varphi = KGM/c^2r,\ K = \text{const., for } \varphi \to 0. \tag{35}$$

Consider now the case of an extended deflecting body, that we shall assume to be spherically symmetric. Now the problem is much more complicated, as the deflection will also depend on R, the radius of the body, and on the dimensionless function $\lambda(\xi)$ that describes its density distribution as a function of $\xi = s/R$ (s is the distance from the center). Then if one considers a class of bodies having the same structure, i.e., with the same density distribution $\lambda(\xi)$, the deflection will depend on φ and on $\zeta = R/r$:

$$\theta = \Phi_\lambda(\varphi, \zeta), \tag{36}$$

where the function Φ_λ will depend on the density distribution of the bodies, as described by λ.

As before, it is reasonable to assume that for a sufficiently small mass, i.e., for $\varphi \to 0$, the deflection is arbitrarily small, so that $\lim_{\varphi\to 0}\Phi_\lambda = 0$, and that $\lim_{\varphi\to 0}(d\Phi_\lambda/d\varphi) \equiv \psi_\lambda(\zeta)/\zeta$ exists. Then

$$\theta \cong \varphi\psi_\lambda(\zeta)\zeta = (GM/c^2R)\psi_\lambda(\zeta) \quad \text{for } \varphi \to 0. \tag{37}$$

In (37), $\psi_\lambda(\zeta)$ must tend to $K\zeta$ as $R \to 0$, i.e., $\zeta \to 0$. Then it seems reasonable to assume $\psi_\lambda(\zeta) \cong K\zeta$, so that

$$\theta \cong K(GM/c^2R)(R/r) \quad \text{for } \varphi \to 0. \tag{38}$$

For the case of the Sun, $\varphi \leqslant (GM/c^2R) \approx 2 \times 10^{-6} \approx 0.4'' \ll 1$, so that it can be conjectured that the approximate formula (38) can be applied. One then obtains:

$$\vartheta \approx 0.4'' \times K(R/r). \tag{39}$$

The magnitude of the dimensionless proportionality factor K cannot be found by dimensional analysis. Its determination requires a more detailed physical theory (relativity). Nevertheless, its order of magnitude may be expected *a priori* to be unity; the relativistic value is 4.

2.5.2 *The advance of the perihelion of Mercury*

Let us assume that, to a first approximation, a planetary orbit is a closed curve about the Sun. Consider also a class of geometrically similar orbits. Then the period P is a function only of a parameter related to the size of the orbit, such as its major half axis a (see Figure 12), of the mass of the Sun, M, and of G (Kepler's Third Law):

$$P = C(a^3/GM)^{1/2}, \tag{40}$$

where the constant C depends only on the shape of the orbit. If the mass m of the planet is small but not negligible, this formula can be generalized to

$$P = C(a^3/GM)^{1/2}f(m/M). \tag{41}$$

Now imagine that as a better approximation, the planetary motion is obtained by a superposition of the motion around the closed orbit, with a slow rotation of the orbit in its plane. Let α be the angle through which the closed curve rotates during one revolution of the planet.

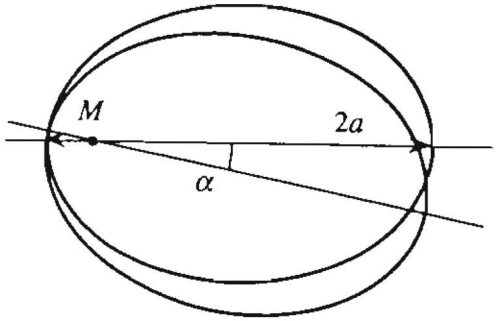

FIGURE 12 Advance of the perihelion.

Since this precession is a relativistic effect, α will be determined by c, in addition to G, M, and a. Therefore:

$$\alpha = \Psi(\psi), \psi = GM/c^2a. \tag{42}$$

It is reasonable to assume that $\lim_{\psi\to 0}\Psi(\psi) = 0$, and that $\lim_{\psi\to 0} d\Psi/d\psi = K \neq 0$. In consequence, for small ψ we obtain, with good approximation

$$\alpha \cong KGM/c^2a. \tag{43}$$

The angular velocity of the precession is then

$$\omega = \alpha/P \cong k(G^3M^3/c^4a^5)^{1/2}, \tag{44}$$

in which the proportionality factor k depends on the shape of the orbit, but can be expected to be of the order of unity.

A large collection of applications of dimensional analysis to astrophysical problems can be found in Kurth (1972); these include topics of interest for stellar structure such as polytropic and isothermal gas balls, stellar envelopes, stellar interiors, etc. In the classical monograph of Sedov (1959) the reader can find applications to the equilibrium and motion of a self gravitating gaseous mass as well as mass, radius and luminosity relations, equilibrium solutions, pulsating stars and novas and supernovas. Impact cratering is discussed in Zel'dovich and Raizer (1967).

3 THE CONCEPT OF SELF SIMILARITY

There is an important class of phenomena in which scale symmetry allows to reduce the number of independent variables of the problem. This happens because the solution is similar to itself (self similar) if the variables are conveniently scaled. In gas dynamics, fluid mechanics, in the physics of waves, as well as in many other fields of physics, one can find many instances of self similarity. Let us discuss a couple of examples that bring into evidence its basic features.

3.1 The Diffusion of Heat

We shall present here this problem, although it does not belong to the dynamics of fluids, because is one of the simplest examples of self similarity (Barenblatt, 1979), and in addition is mathematically equivalent to the phenomenon of the diffusion of vorticity due to the effect of viscosity (see Sedov, 1959), as well as to (linear) diffusion.

Suppose that at a certain moment, which we shall take as $t = 0$, a certain quantity of heat Q is dumped in a small volume δV of an infinite, homogeneous, and isotropic medium. We shall use a coordinate system with its origin within δV. As time passes, heat will diffuse through the medium. We want to find the temperature distribution $T(r, t)$ for large values of r and t, so that the details of the initial distribution of heat within δV are irrelevant. The variables and parameters of this problem are then

$$T, r, t, \kappa, H = Q/\rho C_p, \tag{45}$$

where κ is the thermal diffusion coefficient, ρ the density, C_p the specific heat capacity, and H $(=\delta V\langle T_0\rangle)$ is related to the initial average temperature within δV. Since the dimensions of all these variables and parameters can be expressed in terms of the fundamental dimensions of length, time, and temperature, there will be two independent scale invariants (dimensionless combinations of the variables and parameters). These invariants can be chosen as:

$$\xi = [r^2/(\kappa t)]^{1/2}, \tau = (\kappa t)^{3/2}H^{-1}T, \tag{46}$$

and therefore one concludes that,

$$\tau = f(\xi) \text{ i.e., } T = \frac{H}{(\kappa t)^{3/2}} f\left(\frac{r}{(\kappa t)^{1/2}}\right). \tag{47}$$

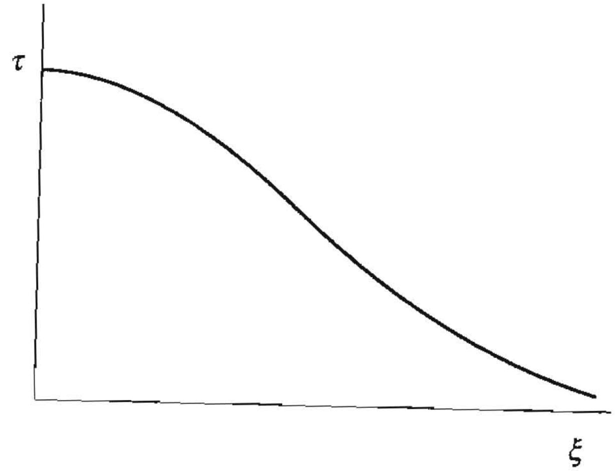

FIGURE 13 Diffusion of heat.

This result means that if r is scaled as $t^{1/2}$, and T as $t^{-3/2}$, the temperature distribution will look the same at any time. In other words, if we represent $T(r, t)$ for any fixed t, the same graph will also represent the temperature distribution $T'(r', t')$ for any other fixed t', provided the scales of the r axis and of the T axis are changed by the scale factors $(t'/t)^{1/2}$ and $(t'/t)^{-3/2}$, respectively (see Figure 13). For this reason we say that the temperature distribution is self similar, that is, similar to itself; in fact, given the distribution at a certain time, the distribution at any other time can be obtained from the first by means of a similarity mapping.

To complete the solution we must determine f. To this end dimensional analysis does not suffice, what is needed is to solve the heat diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \Delta T, \tag{48}$$

(here Δ denotes the Laplacian) subject to the initial and boundary conditions of the problem. Notice that actually in the present case it is not possible to carry out this program, as we have not specified completely the initial temperature distribution. However, if δV is small, and if we are only interested in the solution for large r and t, we can assume that for fixed time, the temperature distribution depends only on the

distance to the origin, then $T(\mathbf{r}, t) = T(r, t)$, and we must solve

$$\frac{\partial T}{\partial t} = \kappa r^{-2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right). \tag{49}$$

The solution of the partial differential equation (49) is still a complicated affair. Here is when self similarity comes to our help, since $f(\xi)$ does not depend on r and t in an arbitrary way, but only through the combination

$$\xi = \frac{r}{(\kappa t)^{1/2}}. \tag{50}$$

Thanks to this fact an important simplification is achieved, because actually it is sufficient to solve an ordinary differential equation in the single independent variable ξ, that is called the self similarity variable. The consequence of self similarity is then a reduction of the number of the independent variables of the problem: in the present case we pass from the two variables r and t to the single variable ξ. Substituting (46) and (47) in (49), we find an equation for f:

$$\xi(2f'/\xi + f)' + 3(2f'/\xi + f) = 0, \tag{51}$$

in which $'$ denotes the derivative with respect to ξ. Since the solution we need must vanish at infinity, we must have

$$2f'/\xi + f = 0, \tag{52}$$

that can be immediately integrated yielding

$$f = Ke^{-\xi^2/4}, \tag{53}$$

where K denotes a normalization factor, so that one finally obtains

$$T(r, t) = \frac{KH}{(\kappa t)^{3/2}} e^{-r^2/4\kappa t} \quad (K = \pi^{-3/2}). \tag{54}$$

We notice that (51) is an exact solution of (49) that describes for any time the temperature distribution produced by an instantaneous point source of heat, i.e., a source whose spatial distribution is a Dirac delta function. However, its meaning goes far beyond that, because whatever initial temperature distribution we assume (as long as it is localized within the small volume element δV), it will always tend for large r and t (that is, asymptotically) to the self similar solution (54). Many

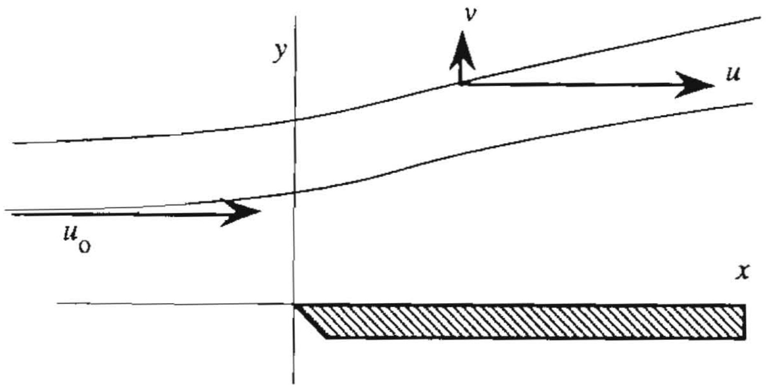

FIGURE 14 Geometry of the laminar boundary layer problem.

phenomena share this property, of tending asymptotically to self similarity when certain details of the initial conditions can be ignored because they have ceased to be relevant.

3.2 The Laminar Boundary Layer

Let us consider the steady flow of a viscous incompressible fluid over a semi infinite plane plate located at $y = 0$ and extending on the interval $0 < x < \infty$ (see Figure 14). Let y be the coordinate perpendicular to the plate. We assume that the fluid is moving in the positive x direction, and that for $x \ll 0$ its velocity u_0 is uniform and parallel to the x axis (see Sedov, 1959). The fluid occupies all the space beyond the plate. We shall study this problem by means of the equations of Prandtl (1904), that can be derived from the Navier–Stokes equation by means of some approximations (see for example Landau and Lifschitz, 1959b):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{55}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{56}$$

Here u and v are, respectively, the x and y components of the velocity, and ν is the kinematic viscosity coefficient. To derive these equations it

has been assumed that the thickness d of the boundary layer is small, so that the flow is nearly parallel to the surface (that is, $v \ll u$), and that near the plate the velocity varies rapidly in the y direction, but slowly in the x direction, so that it changes appreciably only over distances much larger than d; in consequence $\partial/\partial y \gg \partial/\partial x$. The boundary conditions of the problem are

$$u = v = 0 \text{ at } y = 0,\ x > 0, \tag{57}$$

$$u = u_0 \text{ at } y = \infty. \tag{58}$$

The parameters are

$$u_0,\ \nu,\ x,\ y, \tag{59}$$

and there is no characteristic length (the plate is infinite). The dimensionless parameters are then

$$\eta = \frac{y}{x},\ \zeta = \frac{y}{\sqrt{\nu x/u_0}}. \tag{60}$$

The solution must then be of the form

$$u = u_0 f(\eta, \zeta),\ v = \sqrt{\frac{\nu u_0}{x}}\,\Phi(\eta, \zeta). \tag{61}$$

We shall now show that the parameter η is not essential. To this purpose let us perform the following change of variables:

$$x = lX,\ y = \sqrt{\frac{\nu l}{u_0}}\,Y,\ u = u_0 U,\ v = \sqrt{\frac{\nu u_0}{l}}\,V, \tag{62}$$

where l is a constant length (> 0), so that the new variables X, Y, U, V are dimensionless (notice that this transformation employs different scales for the variables x and y: it is then an affinity). The equations (55) and (56) are transformed into

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2},\quad \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{63}$$

and the boundary conditions are

$$U = V = 0 \text{ at } Y = 0,\ X > 0, \tag{64}$$

$$U = 1 \text{ at } Y = \infty. \tag{65}$$

According to (61) we must have

$$U = f\left(\frac{Y}{X\sqrt{\mathscr{R}}}, \frac{Y}{\sqrt{X}}\right), \; V = \sqrt{X}\Phi\left(\frac{Y}{X\sqrt{\mathscr{R}}}, \frac{Y}{\sqrt{X}}\right). \tag{66}$$

where $\mathscr{R} = u_0 l/\nu$ is the Reynolds number. These equations, together with the boundary conditions (64) and (65) are the dimensionless formulation of the problem of the boundary layer. We now observe that the solution we are seeking cannot depend on $\mathscr{R}$, since this parameter does not appear in the Prandtl equations (63) nor in the boundary conditions. Then it is clear that the first argument, y/x, cannot enter in the formulae (61), and the solution must be of the form

$$u = u_0 f(\zeta), \; v = \sqrt{\frac{\nu u_0}{x}}\,\Phi(\zeta), \; \zeta = \frac{y}{\sqrt{\nu x/u_0}}, \tag{67}$$

so that it is self similar in the variable ζ. It is interesting to notice that in this case the self similarity does not follow only from scale invariance (whose consequence is (61)), but from the additional fact that the equations of Prandtl are invariant with respect to a larger transformation group, that includes the affinity (62). We shall come back to this issue later on.

From (67) we can derive a scaling law for the thickness of the boundary layer. We obtain:

$$d = y(\zeta^*) = \zeta^*\sqrt{\nu x/u_0}, \tag{68}$$

where ζ^* is the value of ζ for which u attains a certain fraction of its asymptotic value u_0 (see Figure 15).

To complete the solution of the problem it is necessary to find f and Φ. To this end we change the dependent variable f according to:

$$f = \phi'(\zeta). \tag{69}$$

From the continuity equation we obtain

$$\Phi' = \tfrac{1}{2}\zeta\phi'' = \tfrac{1}{2}(\zeta\phi' - \phi)'. \tag{70}$$

Then, using this relationship

$$u = u_0\phi', \quad v = \sqrt{\frac{\nu u_0}{x}}\,\frac{1}{2}(\zeta\phi' - \phi), \tag{71}$$

and substituting in (55) we get

$$2\phi''' + \phi\phi'' = 0. \tag{72}$$

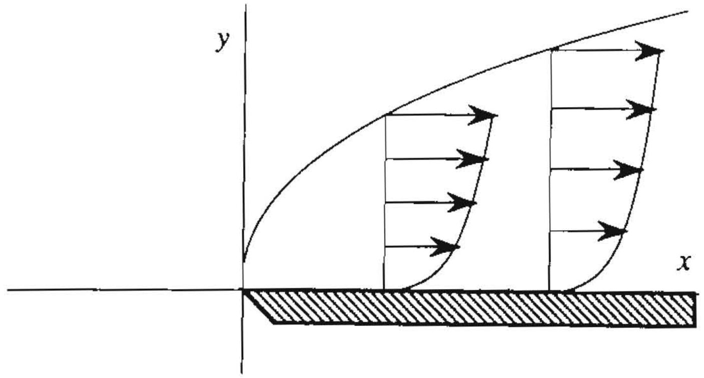

FIGURE 15 The thickness of the boundary layer.

The boundary conditions take the form:

$$\phi'(0) = \phi(0) = 0, \; \phi'(\infty) = 1. \tag{73}$$

The nonlinear differential equation (72) with the boundary conditions (73) can be solved numerically (Blasius, 1908). It is convenient to take advantage of the following very general property of (72) (Töpfer, 1912): if ϕ_0 is a solution of (72), then for any constant a

$$\phi(\zeta) = a\phi_0(a\zeta), \tag{74}$$

is also a solution. Then, let us call ϕ_0 the solution of (72) that satisfies the boundary conditions

$$\phi_0(0) = \phi_0'(0) = 0, \; \phi_0''(0) = 1. \tag{75}$$

This solution can be found by means of the usual methods of numerical integration. From it one can evaluate the limit

$$\lim_{\zeta\to\infty}\phi_0'(\zeta) = k(=2.0854\ldots). \tag{76}$$

We next set

$$\phi(\zeta) = \alpha^{1/3}\phi_0(\alpha^{1/3}\zeta), \tag{77}$$

that satisfies the boundary conditions

$$\phi(0) = \phi'(0) = 0, \; \phi''(0) = \alpha; \; \lim_{\zeta\to\infty}\phi'(\zeta) = k\alpha^{2/3}. \tag{78}$$

Then, with $\alpha = k^{-3/2}(=0.332\ldots)$ we obtain the desired solution, that satisfies at infinity the condition (73).

The preceding examples illustrate the advantages that derive from the self similarity of a phenomenon. Self similarity simplifies the analysis and the representation of the characteristics of the problem. In the above examples, the self similarity of the solutions of the governing partial differential equations allowed to reduce the latter to ordinary differential equations, thus making the mathematics much simpler.

The widespread use of computers produced a change of disposition with respect to self similar solutions, but they did not cease to deserve interest. Before the advent of computers, the possibility of reducing partial differential equations to ordinary differential equations was considered very important, so that the interest in the self similar solutions was mainly due to the fact that they are easy to obtain and analyze. Then the situation changed, as in many problems it was found that the simplest procedure for solving numerically the boundary value problems for the systems of ordinary differential equations that arise in the construction of self similar solutions, is to solve the original partial differential equations by means of stabilization methods. Nevertheless, self similarity still attracts much attention because it is a manifestation of a deep physical property that consists in the presence of a certain type of stabilization in the phenomenon under study. In addition, self similar solutions are used as a starting point for numerical calculations with computers, and as a comparison standard to check approximate methods for the solution of more complex problems.

4 SELF SIMILARITY AND INTERMEDIATE ASYMPTOTICS

Superficially self similarities seem to be nothing else than isolated exact solutions of certain special problems, perhaps elegant, sometimes useful, but of a limited scope and significance in what concerns the fundamental properties of physical theories. It took a deeper understanding to realize that the meaning of this type of solutions goes much beyond that of being a simple description of the behavior of systems under very particular conditions. Actually the self similar solutions reveal the intermediate asymptotic behavior of the solutions of a much wider class of phenomena. By intermediate asymptotics we mean the range in which these solutions have ceased to depend on certain details of the initial and/or the boundary conditions, although the system is still far from having arrived to its limiting state.

Take for example the diffusion of heat: we have seen that the solution (54) not only describes the temperature distribution due to an instantaneous point source; it also describes the temperature distribution in a finite region, of a certain size Λ, as long as initially the same quantity of heat is concentrated not at a single point, but in a finite region δV (it is not even necessary that this region be symmetric), whose linear size λ is such that[†]

$$\lambda <<< \Lambda. \tag{79}$$

The temperature is measured at a distance r from the center of δV, such that $\lambda \ll r$, and at the same time $r \ll \Lambda$ (Λ can be thought as the distance of the boundaries). This intermediate asymptotic property of the solution (54) can be rigorously derived in the mathematical theory of heat conduction.

Similar comments can be made concerning the laminar boundary layer, since actually the property we are considering is quite general. The self similar solutions are always solutions of degenerate problems in which the constant parameters whose dimensions are the same as those of the independent variables of the problem, take values which are zero or infinite. Accordingly the self similar solutions always correspond to singular initial and/or boundary conditions. This is what happens in the examples of the preceding Section. Then, the self similar solutions always represent the intermediate asymptotics of the solutions of non degenerate, non self similar problems (more precisely, the stable self similar solutions, see Barenblatt, 1979).

It is frequently believed that the self similar solutions can be derived from dimensional analysis (i.e., from physical similarity), which if applied to the formulation of a degenerate problem that admits self similar solutions, always allows to obtain the form of the solutions (that is to say the expression of the self similar variables); after the exact self similar solution has been found it is not difficult to find the class of non degenerate problems whose intermediate asymptotics it describes. This is indeed the case of the preceding examples. However, it is essential to recognize that the self similar solutions obtained by means of dimensional analysis do not exhaust the field of self similarities. Actually, it can be shown that many problems have a self similar intermediate

[†]We say that $a <<< b$ when there is a range of values of x such that $a \ll x$, and at the same time $x \ll b$.

asymptotics that cannot be obtained by means of simple dimensional arguments based on the original formulation of the degenerate problem. This is related to the fact that the passage to the limit from the complete (non degenerate, non self similar) problem to its self similar intermediate asymptotics, is not regular.

The interpretation of self similarities as intermediate asymptotics allows to clarify the role (and the limitations) of dimensional analysis with respect to establishing self similarity and determining the similarity variables; to this purpose we shall now discuss examples in which dimensional analysis does not allow to achieve these purposes. It was Zel'dovich (1956) who first established on this basis a classification of self similarities in two classes, that he called First and Second Kind. Self similarities of the First Kind are those that are established, and the self similar variables determined, by means of dimensional analysis. The self similarities of the Second Kind are those for which this is not possible.

4.1 Flow Past a Wedge: Self Similarity of the Second Kind

One of the simplest examples of self similarity of the second kind is the well known problem of the plane potential flow of an incompressible fluid around a wedge-shaped obstacle (Barenblatt and Zel'dovich, 1972). The geometry is represented in Figure 16; the cross section of the wedge has the shape of an isosceles triangle, 2α is the angle of the vertex; at infinity, upstream from the wedge, the fluid velocity is parallel to the axis of symmetry of the triangle and its value is U. From dimensional analysis it is evident that the velocity potential $\phi(r, \theta)$ can be expressed as

$$\phi = rUf(\theta, \eta),\ \eta = L/r, \tag{80}$$

in which θ is the polar angle, r the distance to the vertex of the wedge, L its thickness, and f is a dimensionless function of its dimensionless arguments.

We are interested in the limit $L \to \infty$, that is, in the degenerate problem of the flow past an infinite wedge (or, equivalently, in the asymptotics of the solution near the vertex of the wedge). At first sight, it would seem that in this limit the parameter L ceases to be significant,

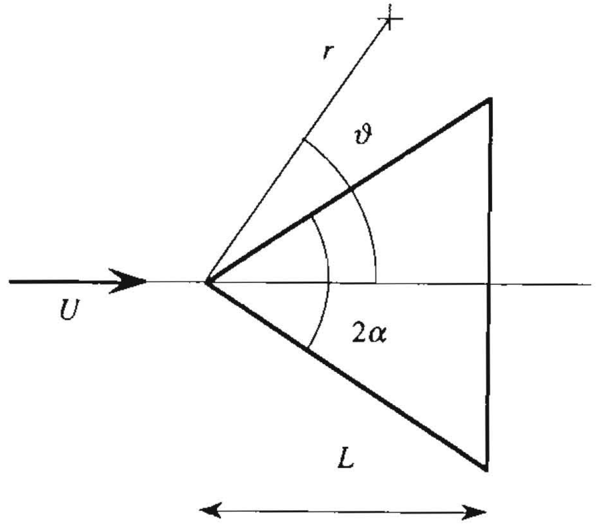

FIGURE 16 Flow past a wedge.

and in consequence the solution should not depend on the second argument of f, that should drop out from the problem. This is not true; in fact it can be easily verified that $\phi = rUg(\theta)$ is not a solution. Now, in this simple problem it is easy to find by means of a conformal mapping the complete non self similar solution in closed form (see for example Landau and Lifschitz, 1959b). It is found that for large η,

$$f(\theta, \eta) = \eta^{\lambda}\phi(\theta) + \text{small quantities}, \tag{81}$$

so that the leading term of the asymptotic expansion of the velocity potential near the vertex, that is by itself a solution of the Laplace equation, has the form

$$\phi = Ar^{1-\lambda}\phi(\theta),\ A = UL^{\lambda}. \tag{82}$$

The value of λ can be found by substituting (82) in the Laplace equation, and requiring that the azimuthal component of the velocity vanishes on $\theta = \pm\alpha$ and $\theta = \pm\pi$, and only on these lines. Thus one finds

$$\lambda = -\alpha/(\pi - \alpha). \tag{83}$$

The solution (82)–(83) is manifestly self similar, but it is evident that it cannot be derived from (80) by passing to the limit $L \to \infty$; in fact, however large is L/r (that is, however near to the vertex), we cannot ignore the second argument of the function f because for $\eta \to \infty$ the function f does not tend to a finite limit. Then η remain essential,[†] however large is L. In addition, the exponent λ cannot be obtained from dimensional analysis, but it is obtained in the course of the construction of the self similar solution[††]. Finally, the numerical value of the constant A cannot be obtained by considering only the self similarity of the solution, but must be found from the analysis of the asymptotics of the full non self similar solution.

4.2 The Modified Heat Diffusion Problem

As a second example of self similarity of the second kind we shall discuss a modified version of the problem of Section 3.1; the difference is that now we shall assume that the thermal diffusion coefficient κ has a constant value when the medium is heating up, but a different value κ', also constant, when it is cooling off (see Barenblatt, 1979). This is what happens, for example, if pores are produced in the medium when it cools. The same type of equtions are also found in the theory of filtration of an elastic fluid in an elasto-plastic porous medium (Barenblatt and Krylov, 1955, Barenblatt *et al.*, 1972). We shall consider the case of planar symmetry. The equations of the problem are

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}\left(\frac{\partial T}{\partial t} \geqslant 0\right), \quad \frac{\partial T}{\partial t} = \kappa' \frac{\partial^2 T}{\partial x^2}\left(\frac{\partial T}{\partial t} < 0\right), \tag{84}$$

in place of (49). It is essential that the discontinuous behavior of the thermal diffusion coefficient be related to a difference of the specific heat capacity between heating and cooling, and that the thermal conductivity be independent of the sign of the temperature change. Then $\partial T/\partial x$ must be continuous to ensure the continuity of the heat flow.

We shall now try to construct a solution representing the effect of the instantaneous deposition of a quantity of heat at the origin. This solution must satisfy the following condition at $t = 0$ and at $x = \infty$:

$$T(x, 0) \equiv 0 \ (x \neq 0); \int_{-\infty}^{\infty} T(x, 0)\, dx = H; \ T(\infty, t) \equiv 0 \tag{85}$$

[†]We say that a parameter is essential when it actually governs the phenomenon.

[††]It is evident that it is not possible, by dimensional analysis alone, to determine any dimensionless parameter other than the exponents of the physical quantities that enter in the expressions of the invariants of the problem, according to the Pi theorem.

in which H is a constant. It is well known (and it was shown in Section 3.1 for the spherically symmetric case) that in the case $\kappa = \kappa'$ such a solution exists, and is self similar. It would seem that the same dimensional considerations of the former case should also hold when $\kappa \neq \kappa'$. In effect, the modification we have introduced has increased the number of parameters, but the only difference is that now the new dimensionless parameter $\varepsilon = \kappa'/\kappa$ has appeared. Then, it should be expected that the solution we are seeking might be expressed in the form

$$T = \frac{H}{\sqrt{\kappa t}} \Phi(\xi, \varepsilon), \ \xi = \frac{x}{\sqrt{\kappa t}}, \tag{86}$$

where Φ is continuous with a continuous derivative with respect to ξ, and is an even function. Furthermore, by virtue of the self similarity, the domain in which T increases must be given by

$$|x| \geqslant x_0(t) \equiv \xi_0\sqrt{\kappa t}, \tag{87}$$

where ξ_0 is a constant that depends on ε.

However it is not difficult to verify (we omit details for brevity, see Barenblatt, 1979) that if $\kappa \neq \kappa'$ there is no solution of (84) of the form (86), that is continuous, has a continuous derivative with respect to x (i.e., has a continuous heat flow), is even, and vanishes at infinity.

The paradox can be solved if we observe that the conditions (85) are of a singular character, and must be described by a generalized function (a distribution). The solution that satisfies these conditions, if it exists, should represent the asymptotics for large times of the class of solutions that satisfy initial conditions described by ordinary smooth functions (i.e. functions that are continuous and that have continuous derivatives up to a required order) of the form (see Figure 17):

$$T(x, 0) = \frac{H}{L} T_0\left(\frac{x}{L}\right), \tag{88}$$

where L is a certain length that measures the size of the region where heat was initially deposited, and T_0 is an even, smooth dimensionless function that decreases monotonically, faster than any power, as the absolute value of its argument is increased, and that in addition satisfies obvious normalization conditions. It can be shown (Kamenomostskaya, 1957) that with these initial conditions the solution of the Cauchy problem exists, is unique, and satisfies the remaining conditions we are requiring. However, the new dimensional parameter L enters in

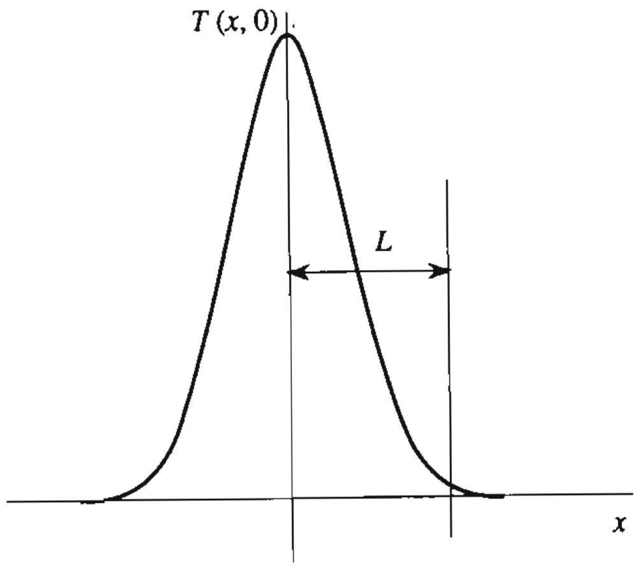

FIGURE 17 Initial condition for the modified heat diffusion problem.

the problem, so that the solution is no longer self similar. In fact, dimensional analysis tells us that now

$$T = \frac{H}{\sqrt{\kappa t}}\Phi(\xi, \eta, \varepsilon), \eta = \frac{L}{\sqrt{\kappa t}}. \tag{89}$$

The exact self similar solution for the instantaneous source of the case $\kappa = \kappa'$ corresponds to the singular initial condition that results from (88) when $L \to 0$. But this solution represents in addition something else. We notice that (89) is also valid for $\varepsilon = 1$, and that $\eta \to 0$ when $t \to \infty$. By an adequate choice of x we can pass to this limit in such a way that

$$\xi = \frac{x}{\sqrt{\kappa t}} = \text{const.}, \tag{90}$$

and in the limit we obtain the already mentioned self similar solution. For this reason, as said before, when $\kappa = \kappa'$ the self similar solution of the singular initial conditions problem is, in addition, the asymptotics for large times of a whole class of solutions of regular initial value problems. The non existence, for $\kappa \neq \kappa'$, of a solution of the singular initial values problem indicates that now, when $\eta \to 0$, the function $\Phi(\xi, \eta, \varepsilon)$ does not tend to a finite, non vanishing limit. But notwithstanding this, the solution (89) has still a self similar asymptotics. In fact it can be shown that a real number $\alpha = \alpha(\varepsilon)$ exists, such that the limit

$$\lim_{\eta \to 0} \eta^{-\alpha}\Phi(\zeta, \eta, \varepsilon) = \phi(\zeta, \varepsilon), \tag{91}$$

exists, and is finite and non vanishing. Then, for $\eta \to 0$, Φ admits the asymptotic representation

$$\Phi(\zeta, \eta, \varepsilon) = \eta^{\alpha}\phi(\zeta, \varepsilon) + \text{small quantities}. \tag{92}$$

In consequence the asymptotics of our problem, for $t \to \infty$, is not expressed by (86), but is of the self similar form

$$T = \frac{HL^{\alpha}}{(\kappa t)^{(1+\alpha)/2}}\phi(\xi, \varepsilon). \tag{93}$$

Notice that the passage to the limit $\eta \to 0$ for finite ξ can be also effected by taking the limit $L \to 0$ with fixed x, t (in the case $\kappa = \kappa'$, i.e., $\varepsilon = 1$, this leads to the instantaneous point source solution). But according to (93) this limit, for fixed H and $\alpha \neq 0$, is zero or infinity, according if α is positive or negative. Then in passing to the limit $L \to 0$ with x, t fixed, it is necessary (for $\alpha \neq 0$) that simultaneously $H \to \infty, 0$ (according to the sign of α), in such a way as to ensure that the product in the numerator of (93) maintains a finite value; only then we can obtain the same limit as was obtained for finite L and $t \to \theta$.† The solution resulting from this passage to the limit is self similar, but is not of the form (86); what we get is:

$$T = \frac{A}{(\kappa t)^{(1+\alpha)/2}}\phi(\xi, \varepsilon), \; A = \beta \lim_{L\to 0} HL^{\alpha}, \; x_0(t) \equiv \xi_0\sqrt{\kappa t}. \tag{94}$$

Here β is a dimensionless constant that depends on the normalization of ϕ, and the quantities α and A are what remains of the parameters H and L after passing to the limit. To evaluate them two methods can be followed:

(a) First, we can compute numerically the non self similar solution

†The reader should not be confused by this apparently complicated affair of passing to a limit in which $L \to 0$, $H \to \infty$, 0, etc.. It is nothing else than a formal mathematical trick that allows to find the degenerate problem whose solution is the correct asymptotics of the real, non degenerate problem in which, of course, L and H have finite, constant values.

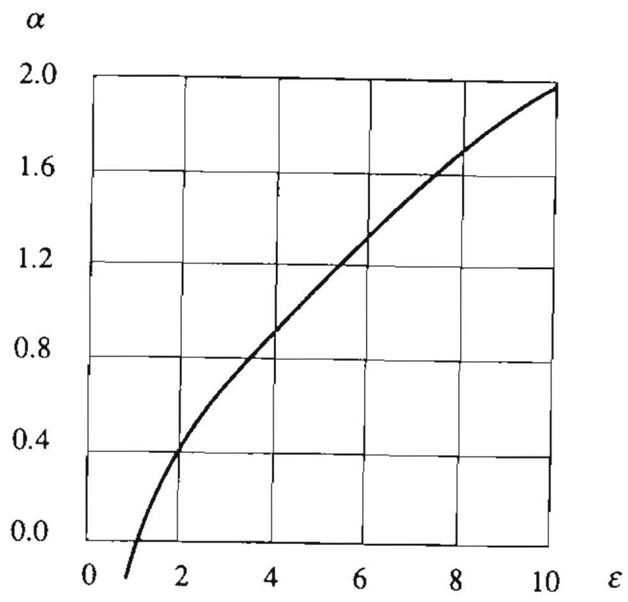

FIGURE 18 The eigenvalue α as a function of ε.

(89) and follow its evolution until it passes into the self similar asymptotics (this has been done by V. M. Uroev, see Barenblatt and Sivashinskii, 1969, and it was observed that the solution tends rather rapidly to the asymptotics given by (94)).

(b) Second, it is also possible to construct directly the self similar solution by taking (94) as an ansatz, and substituting it in the governing equations and the initial conditions; in this case α and A are unknowns of the problem, and are determined by requiring the existence of the solution at large; this procedure leads to a nonlinear eigenvalue problem for α, that for brevity we shall not present here in detail (the interested reader can find the complete analysis in the monograph of Barenblatt, 1979). The result is shown in Figure 18.

It must be noticed that the last method of solution does not allow to determine completely the self similar asymptotics, since the constant A (or, equivalently, the dimensionless constant β) is obtained within a

constant numerical factor.[†] In the classical case ($\varepsilon = 1$) in which $\alpha = 0$, the value of this constant can be obtained from the conservation law

$$\int_{-\infty}^{\infty} T(x, t)\, dx = H, \tag{95}$$

however this law is not valid if $\kappa \neq \kappa'$ (see Barenblatt, 1979) since A is a complicated functional of the initial temperature distribution, and cannot be obtained from the initial conditions by application of conservation laws. Notice also that the self similar asymptotics we have found is no longer a solution of the instantaneous point source problem, it is the solution of a different degenerate problem, namely that which is obtained in the limit $L \to 0$ by assuming that the quantity of heat ($\sim H$) that must be deposited initially in the region of size L varies as L decreases. This is necessary in order to arrive to the same asymptotic representation for large t as that of the solution of the original non degenerate problem: H must increase if $\varepsilon > 1$ and decrease if $\varepsilon < 1$, so as to keep A constant. Finally it should be observed that according to the solution (94), the variation of the temperature in the point where it is a maximum, and the position of the point where the thermal diffusivity is dicontinuous are given by

$$T_{\max} \sim \frac{A}{(\kappa t)^{(1+\alpha)/2}}, \quad x_0(t) = \xi_0\sqrt{\kappa t}. \tag{96}$$

The second of these formulae can be easily found from similarity arguments, starting from the concept of an instantaneous point source. The first one, on the contrary, is impossible to derive by means of this type of argument, notwithstanding that the scaling follows a power law, and is completely determined by the dimensions of A. What happens is that the dimensions of A are not known in advance: it is first necessary to find α by solving the eigenvalue problem.

The previous examples illustrate the main features of the self similar solutions of the second kind. We observe that:

[†]The origin of this indeterminacy is that the same degenerate problem that is solved by direct construction must give the asymptotics of an inifinite set of non degenerate problems in which L and H have different finite, constant values.

(a) It is not possible, by dimensional analysis alone, to construct the self similar solution of the degenerate problem.

(b) Nevertheless, the complete (non degenerate) problem has a self similar intermediate asymptotics. The impossibility of finding it by means of dimensional analysis can be traced to the fact that the passage to the limit in which some of the parameters of the non degenerate problem tend to zero or to infinity, is not regular.

(c) In the self similar asymptotics, one of the independent variables appears with an exponent that is not known in advance (and that by principle we cannot evaluate by means of dimensional analysis).

(d) This exponent can be determined by direct construction of the solution, and is obtained during the process of finding the self similar asymptotics, as the solution of a nonlinear eigenvalue problem.

4.3 Complete and Incomplete Self Similarity

We have found in discussing the preceding examples that the self similarities can be classified in two groups:

(a) In some instances there is a complete statement of the degenerate problem, and by the application of dimensional analysis in the usual manner plus eventually some additional symmetry consideration, we can establish the self similarity of the solution and construct the self similar variables. In addition, if we succeed in evaluating some integral, we can obtain the solution in closed, finite form. This is what happens for the instantaneous point source of heat (Section 3.1), and for the laminar boundary layer (Section 3.2).

(b) However, it may be sufficient to modify slightly the problem (for instance in the case of heat conduction, to assume that the thermal diffusion coefficient has a certain value when the medium is heating up, and a different value when it is cooling), so slightly that at first sight one is led to believe that the same similarity arguments are still valid, to arrive at contradictions because the modified degenerate problem thus obtained has no valid solution. When this happens, a deeper investigation of the difficulty discloses that the attempt to find solutions by the standard procedure, starting from the degenerate problem, is improperly posed.

It is then convenient to clarify the limitations of dimensional analysis

in connection to the quest for the self similar solutions. To this purpose it is crucial to keep in mind that when we are looking for the self similar solution that represents a certain phenomenon, we are not actually interested in the exact solution of a degenerate problem, but rather in the asymptotics of the solutions of non degenerate problems. There is usually no doubt (for example, because an existence theorem has been demonstrated) that a solution of the non degenerate problem does exist, but when dimensional analysis is applied we of course find that it is not self similar. Which will be the form of the asymptotics of this non self similar problem will depend on the passage to the limit in which the additional parameter (that which spoils the self similarity) tends to zero (or to infinity): in certain cases this limit may be finite and non vanishing, but in others it may be zero, or infinity, or it may not exist at all. In the first case one finds self similarity of the first kind. In the other instances the situation is more complicated: it may lead to self similarity of the second kind, but it may also happen that the problem has no self similar asymptotics at all.

In Sections 4.1 and 4.2 we have shown that in certain situations the above mentioned limit may be zero or infinity, but notwithstanding this the problem admits a meaningful asymptotics, which moreover is self similar. This is precisely the asymptotics we need.

In these cases, the passage to the limit that leads to the self similar asymptotics has certain peculiarities. Consider for instance the example of Section 4.2: in passing to the limit we cannot assume that the amount of heat initially deposited is fixed, and at the same time that the hot region tends to a point. To arrive at the correct asymptotics of the original non degenerate problem it is essential to assume that as the size of the region is changed the amount of heat must increase (or reduce) so that a certain "moment" A of the initial temperature distribution remains constant. Likewise, in the case of the flow past a wedge, we cannot assume from the beginning that the latter is infinite: as the thickness of the wedge is increased we must vary U (the velocity at infinity) so as to keep constant the parameter A. It is essential that the power of L in the expression of these moments is not known in advance, and it is impossible to determine by means of dimensional analysis. This exponent is found in the course of the construction of the solution, as the intermediate asymptotics is derived, either by solving an eigenvalue problem, or by studying numerically the asymptotics of the non degenerate problem.

This is how the two kinds of self similarity arise. One could think that the difference between them is related to whether (or not) the problem admits some integral conservation law that holds also in the non self similar stage. This is not true; in fact we shall show in general that this difference is a consequence of the character of the transition from the non self similar solution to its self similar asymptotics.

To this purpose, consider a non degenerate problem that is governed by n dimensional variables and parameters $a_1, \ldots, a_n$ of which $a_1, \ldots, a_k$ have independent dimensions, and the dimensions of $a_{k+1}, \ldots, a_n$ can be expressed in terms of the dimensions of $a_1, \ldots, a_k$. Let a be any other quantity in which we are interested, that is a function of the governing variables and parameters. According to the Pi theorem any relationship between $n + 1$ dimensional quantities of the form

$$a = f(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n), \tag{97}$$

can be written as

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_{n-k}), \tag{98}$$

where $\Pi, \Pi_1, \ldots, \Pi_{n-k}$ are dimensionless, and

$$\Pi = \frac{a}{a_1^p a_2^q \cdots a_k^r}, \quad \Pi_1 = \frac{a_{k+1}}{a_1^{p_{k+1}} \cdots a_k^{r_{k+1}}}, \ldots, \tag{99}$$

$$\Pi_{n-k} = \frac{a_n}{a_1^{p_n} \cdots a_k^{r_n}}.$$

Let us now consider one of the governing parameters of the problem, say a_{k+1}. Usually this parameter is considered to be essential if the value of the corresponding dimensionless parameter Π_i is neither too large or too small, say, for instance, if its value lies in the range 0.1–10. If Π_i is outside this range it is assumed that its influence in the phenomenon is negligible. Actually this argument is correct only if

$$\lim_{\Pi_i \to 0/\infty} \Phi \text{ exists and is } \neq 0. \tag{100}$$

where in passing to the limit we must keep constant the remaining Π_j.† Of course this needs not to be true in general. However, if such is the case, and if Π_i is sufficiently small (large), we can without loss of precision replace Φ by a function Φ_0 with one argument less:

$$\Pi = \Phi_0(\Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k}), \tag{101}$$

i.e., with $n - k - 1$ arguments. Here

$$\Phi_0 = \lim_{\Pi_i \to 0/\infty} \Phi. \tag{102}$$

In these cases it is said that there is complete self similarity of the phenomenon with respect to the parameter Π_i.

Let us now assume that (100) does not hold, but instead

$$\lim_{\Pi_i \to 0/\infty} \Phi = 0/\infty. \tag{103}$$

It is evident that in this case Π_i remains essential however small (or large) it is, and clearly as $\Pi_i \to 0(\to \infty)$ it is not possible to replace Φ by its limit, because only useless relationships such as $\Pi = 0(= \infty)$ are obtained. Then in the present situation we can not simply cancel Π_i from the list of parameters, and replace f or Φ by functions with one argument less. There is not in this case complete self similarity with respect to Π_i.

Yet there are still some cases in which it is possible to reduce the number of parameters. For instance, suppose some real number α exists, such that when $\Pi_i \to 0(\to \infty)$

$$\Phi = \Pi_i^\alpha \Phi_1(\Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k}) + 0(\Pi_i^\alpha), \tag{104}$$

in which $0(x)$ denotes a quantity that is arbitrarily small as compared to x for x sufficiently small, or large. Then, for sufficiently small (large) Π_i we obtain

$$\Pi^* = \Pi\Pi_i^{-\alpha} = \Phi_1(\Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k}) \tag{105}$$

$$= \frac{a}{a_1^{p-\alpha p_{k+i}} \cdots a_k^{r-\alpha r_{k+i}} a_{k+i}^{\alpha}},$$

where Φ_1 has $n - k - 1$ arguments. Therefore we can again describe the asymptotics in terms of one parameter less, as in the case of complete self similarity, but now: (a) the form of Π^* cannot be obtained from dimensional analysis, because it is necessary to know α, and (b) the argument a_{k+i} enters in Π^* so that it remains essential.

As a second example, imagine that the two parameters Π_i, Π_j are small (large), but when $\Pi_i, \Pi_j \to 0(\to \infty)$ independently, $\Phi \to 0(\to \infty)$, or has no limit. Then Π_i, Π_j remain essential however small (large) they

†To be precise something else is also needed: for $\Pi_i \to 0$ $(\to \infty)$, Φ must converge sufficiently rapidly to its limit, so that when $\Pi_i < 0.1$ (> 10), Φ already differs very little from its limiting value.

are, so that the corresponding dimensional parameters a_{k+i}, a_{k+j} remain essential. Also in this situation an exceptional case may occur in which the list of arguments of Φ can be reduced by one. In fact, suppose that two real numbers α, β exist, such that when $\Pi_i, \Pi_j \to 0(\to\infty)$,

$$\Phi = \Pi_i^\alpha \Phi_2(\Pi_j/\Pi_i^\beta, \Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{j-1}, \Pi_{j+1}, \ldots, \Pi_{n-k}) + 0(\Pi_i^\alpha). \tag{106}$$

Then we obtain asymptotically the relationship

$$\Pi^* = \Phi_2(\Pi^{**}, \Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{j-1}, \Pi_{j+1}, \ldots, \Pi_{n-k}), \tag{107}$$

in which Π^* is given by (105), and

$$\Pi^{**} = \Pi_j\Pi_i^{-\beta} = \frac{a_{k+j}}{a_1^{p_{k+j}-\beta p_{k+i}} \cdots a_k^{r_{k+j}-\beta r_{k+i}} a_{k+i}^\beta}. \tag{108}$$

Again in this case the asymptotics of the problem is given in terms of $n - k - 1$ parameters, but now: (a) the form of Π^* and Π^{**} cannot be obtained from dimensional analysis, as it does not allow to determine α and β, and (b) a_{k+i} appears in Π^* and a_{k+i}, a_{k+j} in Π^{**} so that they remain essential.

In a similar way we can imagine cases in which three or more dimensional parameters tend to zero (or infinity) but there is nevertheless an asymptotics in terms of power laws of these parameters. Therefore, in these exceptional cases, notwithstanding that there is not a complete self similarity with respect to $\Pi_i, \Pi_j, \ldots$, there is again a reduction of the number of arguments of the physical law (that is of Φ). In all these cases we speak of incomplete self similarity with respect of the parameters $\Pi_i, \Pi_j, \ldots,$.

Summarizing, if the value of a certain dimensional parameter is small (large) three possibilities can arise:

(1) If

$$\lim_{\Pi_i\to 0/\infty} \Phi \text{ exists, and is } \neq 0 \text{ and finite,} \tag{109}$$

the corresponding parameters: the dimensional a_{k+i} and the dimensionless Π_i can be excluded from the analysis, and the number of arguments of Φ is decreased by one. All the similarity parameters can be found by means of dimensional analysis. We are in the presence of complete self similarity with respect of the parameter Π_i.

(2) If

$$\lim_{\Pi_i\to 0/\infty} \Phi \text{ is } 0,\ \infty, \text{ or it does not exist,} \tag{110}$$

but one of the above mentioned exceptional cases arises, the number of parameters in Φ can also be reduced by one, but not all the remaining parameters, Π, $\Pi_j, \ldots$ can be obtained from dimensional analysis, and the corresponding $a_{k+i}, \ldots$ are still essential however small (large) they may be. We have incomplete self similarity with respect to Π_i.

(3) If the limit does not exist and the above mentioned exceptions do not occur, there is no power law self similarity with respect to Π_i. In this situation it is not possible to obtain a relationship with a smaller number of parameters.

In the last event it may be useful to further recognize a different special case, that in which for large (small) values of the Π_i, some of these parameters separates, although not according to a power law. In other words, the case in which for these values of the parameters, Φ admits a representation of the form

$$\Phi = \Psi(\Pi_i)\Phi_3 + \text{small quantities}, \tag{111}$$

where Ψ is some function of Π_i that is not a power law, for example a logarithm, and the number of arguments of Φ_3 is less than $n - k$. It can be shown that in this case it is also possible to obtain self similarity, but not of the type we are considering, namely that in which the self similar variable is expressed as a monomial of powers of the variables of the problem (power law self similarity).

One of the difficulties that is encountered when one attempts to find self similarities is that often there is no way to predict to which case does the problem at hand belong; one must then explore the various possibilities in turn, starting with the simpler ones, and compare the results with those of numerical calculations, experiments, or other analytical methods.

The present discussion shows that the assumptions (that are frequently made when looking for self similarities) of considering irrelevant certain parameters that break the degeneracy of the problem, entail in general very strong and potentially dangerous hypotheses. These parameters may be essential, and yet it may be possible to get self similarity. In actual practice, to distinguish between the possible cases of self similarity it is necessary to carry out a deep mathematical investigation, that frequently is not feasible in certain difficult non linear problems. Therefore, when obtaining self similar solutions or similarity laws on the basis of dimensional analysis, one is taking risks and to avoid errors it is strongly recommended to verify (at least by means of numerical calculations) that the solutions or the scaling laws one has found do actually describe the asymptotics of the problem. The situation is enormously more complicated when a mathematical formulation of the problem is not available; in this case one must resort to experiments to verify the basic assumptions.

4.4 Self Similarity of the First and Second Kind

Let us now imagine some physical problem that describes a phenomenon, and that has a unique solution. Let a be an unknown, and $a_1, \ldots, a_n$ the independent variables and parameters that govern the problem determining this unique non self similar solutions. The self similar solutions, on the other hand, are always solutions of degenerate problems that are obtained from the original one when certain parameters a_{k+i}, and the corresponding Π_i tend to zero (or to infinity). These solutions, in addition of being exact solutions of the degenerate problem, are also asymptotics (usually intermediate) of a wider class of non degenerate, non self similar problems, to which the solutions of the latter tend when the said parameter tends to zero (or to infinity). Clearly, if the asymptotics of our problem is self similar, and if the self similar variables are power law monomials, we must be in the presence of one of the two first cases we discussed in Section 4.3. According to which is obtained, it will be self similarity of the first, or of the second kind:

(a) Self similarity of the first kind is found when the passage to the limit is regular, that is, when there is complete self similarity with respect to the parameter that spoils the self similarity of the non degenerate problem. In this case, the expressions of all the self similar variables (both dependent and independent) can be obtained by means of dimensional analysis.

(b) Self similarity of the second kind results when the passage to the limit is irregular, but one of the exceptions discussed in Section 4.3, (2) occurs, so that there is incomplete self similarity with respect to the parameter in question. In this case the expressions of the self similar variables cannot in general be derived by means of dimensional analysis. When the solutions are found by direct construction, the determination of the exponent of the self similarity variable leads to a nonlinear eigenvalue problem and the self similar variable is obtained within a constant numerical factor (like the β factor in the definition of A in eq. (94)). This factor can be evaluated by following the evolution of the solution of the non degenerate problem until its asymptotics is attained, for example, by means of a numerical simulation or by experiment.

If it happens that it is possible to find A by the application of some conservation law, it means that the problem can be reduced to a case of self similarity of the first kind by making an adequate choice of parameters (as was seen in connection with the case of the instantaneous point source of heat).

It is also possible to find self similar solutions that are not of the power law type. These self similarities arise from the special case we discussed in Section 4.3, (2); the solutions that are called limiting to self similar in the literature (Sedov, 1959, Barenblatt, 1979) belong to this class.

4.5 Self Similarity and Groups of Transformations

The connection between dimensional analysis and physical similarity as a consequence of scale symmetry, suggests that self similarity must be closely related to the properties of invariance of the governing equations of the phenomenon. The invariance in which we are interested here is related to the group of scale transformations, i.e., the similarity transformations. A similarity transformation is a change of the governing parameters with independent dimensions, of the type

$$a_1' = A_1 a_1, \ldots, a_k' = A_k a_k. \tag{112}$$

Such a transformation is obtained if we pass from the original system of units of measurement to a new system of the same class. Here the $A_1, \ldots, A_k$ are real positive quantities. The values of the remaining parameters $a, a_{k+1}, \ldots, a_n$ vary according to their dimensions as

$$a' = A_1^p A_2^q \cdots A_k^r a, \text{ etc. .} \tag{113}$$

The transformations (112), (113) form a Lie group with k parameters. The quantities $\Pi, \Pi_1, \ldots, \Pi_{n-k}$ are the invariants of the group.

Thus, the Pi theorem is simply a consequence of the principle of the invariance of the physically meaningful relationships between dimensional quantities of the form (97) with respect to the group of the scale transformations of the parameters with independent dimensions: indeed, granted this invariance, it must be possible to represent all such relationships in terms of the invariants of the group; by necessity, any physically meaningful formulation of a problem (and therefore any solution) must be invariant.

However, we must keep in mind that it may happen that the problem we are considering is invariant with respect to a richer group (i.e., a group larger than that of the scale transformations of the parameters with independent dimensions). Then the number of arguments of the function Φ in the invariant relationship (98), that one obtains by application of the Pi theorem by itself, must be further reduced in accordance with the number of parameters of the supplementary group. This is what happens in the case of the laminar boundary layer (Section 3.2). In these cases the solution may turn out to be self similar, and the self similar variables can be determined taking advantage of the invariance with respect to the supplementary group, even if this self similarity is not a result of dimensional analysis, that exploits the invariance with respect of the group of scale transformations of the quantities with independent dimensions. In this connection we may quote the work of Birkhoff (1960) in which the concept of a generalized inspectional analysis of the equations of mathematical physics is introduced; the idea is to look for groups of transformations that leave invariant the governing equations of a certain phenomenon, and seek the solutions that are invariant with respect to these groups (see also Morgan, 1952).

The algorithms for deriving the maximal transformation group that leaves invariant a certain system of differential equations have been developed by Sophus Lie, and their applications to various problems of physics and mechanics can be found in the books of Ovsyannikov (1962) and Bluman and Cole (1974). Lie group methods allow to investigate systematically the similarity solutions of the first and second types of a given set of partial differential equations, which makes this a very powerful technique for analyzing system evolution. A derivation of the Lie group invariance properties of radiation hydrodynamics equations and their associated similarity solutions has been given by Coggeshall and Axford (1986). General procedures to find new symmetry groups for partial differential equations are given by Bluman *et al.* (1987); in this connection see also the papers of Gaffet (1983, 1985) and of Bluman and Reid (1988).

5 SELF SIMILAR SOLUTIONS IN GAS DYNAMICS

There is very extensive literature dealing with self similar solutions of partial differential equations of interest for the mechanics of continuous media and in particular for the mechanics of fluids. A considerable number of problems that lead to similarity solutions are discussed in the books of Sedov (1959), Zel'dovich and Raizer (1967), Stanyukovich (1960), Barenblatt (1979), as well as in countless papers in specialized journals. With such an abundance, to attempt to give a complete list of references would be a hopeless task. In this paper we shall not try to give full coverage to such a large territory, rather we shall discuss in some detail certain specific problems that lead to interesting families of self similar solutions, emphasizing the methods that allow to investigate systematically each family, and pointing out various typical features of the solutions.

In particular, we shall consider time dependent problems in which the symmetry of the phenomenon allows a description in terms of a single spatial coordinate: that is one dimensional, time dependent problems, like flows with planar symmetry (which depend on a single cartesian coordinate), and axially symmetric or spherically symmetric flows. There will be two independent variables: the time t, and a spatial coordinate that we shall generically denote by x. In these problems self similarity leads to an ordinary differential equation in the self similar variable which is a combination of x and t.

For the unsteady, one dimensional problems of the dynamic of gases, Sedov (1959) and Courant and Friedrichs (1948) developed a powerful formalism (called the phase plane formalism) that permits a systematic investigation of the family of the self similar solutions (for a short introduction of this subject, with examples, see also the Second Edition of the well known textbook by Landau and Lifschitz, 1959b, also Zel'dovich and Raizer, 1967). Other authors have developed similar formalisms to study phenomena governed by equations of a different nature. For example, self similar solutions of the equations of Magnetohydrodynamics have been studied by Zmitrenko and Kurdyumov (1975), Velikovich *et al.* (1985), Liberman and Velikovich (1986), Felber *et al.* (1988a, b). The phase plane formalism for the equations of nonlinear diffusion and related phenomena will be developed in Section 7.

In this Section we shall briefly introduce the basic ideas of the phase plane formalism in gas dynamics, that will be the basis for the discussion of various important applications.

5.1 The Phase Plane Formalism

We shall consider an ideal, non viscous gas, and will neglect heat conduction, so that the evolution of any volume element will be adiabatic. No body forces (like gravity) are acting. The governing equations will then be

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \quad \frac{\partial \rho}{\partial t} + x^{-n}\frac{\partial}{\partial x}(x^n \rho u) = 0, \quad \frac{d}{dt}(p\rho^{-\gamma}) = 0. \tag{114}$$

Here u is the velocity, ρ the density, and p the pressure of the gas: γ is the adiabatic exponent, and n is a geometrical index whose value is 0, 1, 2 for planar, cylindrical and spherical symmetry, respectively. We notice that no constant dimensional parameter appears in these equations, so that the motion will be self similar depending on how many parameters with independent dimension enter in the initial and boundary conditions of the problem. If there are no more than two of these we shall obtain self similarity.

It is useful to rewrite the eqs. (114) in terms of the dependent variables u, ρ, and

$$z = c^2 \equiv \gamma\frac{p}{\rho}. \tag{115}$$

The new dependent variable z is related to the temperature of the gas, or equivalently, to the local value of the speed of sound c. Substituting into (114) and using the notation $g = \ln \rho$, one obtains

$$\frac{\partial g}{\partial t} + u\frac{\partial g}{\partial x} + \frac{\partial u}{\partial x} + n\frac{u}{x} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{z}{\gamma}\frac{\partial g}{\partial x} + \frac{1}{\gamma}\frac{\partial z}{\partial x} = 0,$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)[\ln z + (1-\gamma)g] = 0. \tag{116}$$

The equations (116) show that the density enters into the problem only through its logarithm, so that the governing equations are invariant with respect to scalings of the density.

It is easy to verify (for brevity we omit details that the reader can find in the books of Zel'dovich and Raizer, 1967, or of Sedov, 1959) that the dimensional group of the equations (114), or (116) has similarity solutions of the form

$$u = \frac{\delta x}{t}V(\zeta), \quad \rho = A_\omega G(\zeta)x^{-\omega}, \quad z = \left(\frac{\delta x}{t}\right)^2 Z(\zeta), \quad \zeta = \frac{x}{bt^\delta}, \tag{117}$$

in which the dimensional parameters A_ω and b depend, in general, on the initial and boundary conditions of the problem, and V, G, Z are dimensionless functions of the similarity variable ζ. We observe that there must be always a parameter A_ω that determines the density scale: the density may not be uniform in the initial state: the exponent ω determines the corresponding law of variation with x.

Substituting (117) in (116) one obtains a system of equations that can be written in the following compact form:

$$a_{ij}(V, Z, \gamma)\frac{dF_j}{d\ln\zeta} = d_i(V, Z, \delta, \gamma, \omega, n), \tag{118}$$

in which F_j denotes $(V, \ln G, Z)$, and the a_{ij}, d_i, are functions of V, Z and the remaining parameters:

$$(a_{ij}) \equiv \begin{pmatrix} 1 & V-1 & 0 \\ V-1 & \dfrac{Z}{\gamma} & \dfrac{1}{\gamma} \\ 0 & (\gamma-1)Z & -1 \end{pmatrix}, \tag{119}$$

$$d_1 = -(n+1-\omega)V, \quad d_2 = \frac{\omega-2}{\gamma}Z - V(V-1/\delta),$$

$$d_3 = \frac{Z[2(V-1/\delta) + \omega(\gamma-1)V]}{V-1}.$$

Solving the system of equations (118) we obtain

$$\frac{dV}{d\ln\zeta} = \frac{\Delta_1}{\Delta}, \quad \frac{d\ln G}{d\ln\zeta} = \frac{\Delta_2}{\Delta}, \quad \frac{dZ}{d\ln\zeta} = \frac{\Delta_3}{\Delta}, \tag{120}$$

in which $\Delta = \mathrm{Det}(\alpha_{ij}) = (V-1)^2 - Z$, and Δ_i is the determinant of the matrix that is obtained replacing the i-th colum of (a_{ij}) by the vector d_i.

It is important to notice that the a_{ij}, d_i, and thus Δ, Δ_i do not depend either on G or on ζ. Then, the r.h.s. of (120) are only functions of V and Z. This is not a coincidence, but is a consequence of the invariance of (116) with respect to a change of scale of the density, and of the fact that no constant dimensional parameters appear in it.

The property we have just mentioned has a very important consequence for the analysis, namely that a single autonomous first order ordinary differential equation can be extracted from the system (120):

$$\frac{dZ}{dV} = \frac{\Delta_3(V,Z)}{\Delta_1(V,Z)}. \tag{121}$$

Once (121) has been solved, the remaining equations, which can be written as

$$\frac{d\ln\zeta}{dV} = \frac{\Delta}{\Delta_1}, \quad \frac{d\ln G}{dV} = \frac{\Delta_2}{\Delta_1}, \tag{122}$$

are reduced to quadratures. Actually it is sufficient to evaluate only one of these integrals, since by virtue of the adiabaticity of the motion, it is possible to find an algebraic integral of the (120) of the form $F(V, Z, G, \zeta) = \text{const.}$, in which the constant can be evaluated in terms of the initial and/or boundary conditions.

Then the solution of a self similar problem is essential reduced to the integration (that in general will be numerical) of (121). This is the basis of the method of Sedov (1959) and of Courant and Friedrichs (1948). The variables V, Z are called phase variables, and the solutions of the autonomous differential equation (121) are usually represented as integral curves in the (V, Z) plane, called the phase plane. Actually it is sufficient to consider the $Z > 0$ half plane, as the solutions with negative Z are unphysical.

A single integral curve passes through any regular point of the phase plane. Any integral curve (or piece thereof) represents a self similar flow of some type. All conceivable self similarities of the type (117) described

by the governing equations (116) are represented in the phase plane, so that the formalism is complete. The solution of a given self similar problem, characterized by its particular boundary and initial conditions, is represented in the phase plane by a (several) piece(s) of the appropriate integral curve(s), and must satisfy at its ends the boundary conditions. Each piece represents the flow in a certain domain of the independent variables. If the solution we are seeking is represented by two or more pieces, the flows corresponding to each one of them must be adequately matched at the common boundary of the respective domains. This will be shown when discussing the examples.

To determine which curve (or curves) correspond to the problem under study it is necessary to know the behavior of the solutions in the neighborhood of the singular points of the autonomous equation (121). There are in general 9 singular points in the phase plane (of which 6 at the finite). Their position and nature, and the topology of the integral curves, depends on the parameters δ, γ, ω, and n. We shall not present a detailed analysis of these singularities, as it would be too lengthy and clearly beyond the scope of this paper, but when considering specific problems we shall briefly discuss the relevant singular points. For a detailed (but by no means complete) study of the singularities of (121) see Sedov (1959).

It should be noticed that besides the integral curves, corresponding to each singular point $P \equiv (V_P, Z_P)$ there is an exceptional exact self similar solution of the equations of the dynamics of gases, represented by

$$V = V_P, \; Z = Z_P, \tag{123}$$

the variable ζ being free. The function G is determined by the adiabatic law. For these special solutions the physical variables are given by simple power laws of x and t.

For completeness we should add a remark on the particular form of (117), for $t = 0$. At that time the variables of self similar flows follow simple power laws:

$$u(x, t=0) = u_0 x^{1-\mu}, \quad c(x, t=0) = c_0 x^{1-\mu}, \tag{124}$$
$$\rho(x, t=0) = \rho_0 x^{\omega}, \quad p(x, t=0) = p_0 x^{2(1-\mu)+\omega},$$

with $\mu = 1/\delta$, provided that the limits for $t \to 0$ exist. The constants can be obtained from (117) with $|t| = (x/\zeta)^{\mu}$ in the limit $\zeta \to \infty$.

The advantages of the phase plane formalism are:

(a) It allows to obtain systematically all the self similar solutions of the governing equations.

(b) It is only necessary to integrate a first order equation.

The counterpart of these advantages is that the physical interpretation of the integral curves that are obtained solving (121) is far from obvious, and cannot be appreciated by simply looking at the phase plane; to arrive to it further studies are needed.† Perhaps owing to this obscurity the family of the self similar solutions of (114) has not been yet fully explored, and many integral curves are still waiting to be interpreted.

5.2 Self Similar Unsteady Gas Flows

As said above the family of the self similar one dimensional unsteady flows of a gas is represented in the phase plane by integral curves, solutions of (121). Using the expressions of the Δ_i one can write (121) as:

$$\frac{dZ}{dV} = \frac{Z\{[2(V-\mu)+\nu(\gamma-1)V](V-1)^2-(\gamma-1)V(V-1)(V-\mu) - [2(V-\mu)+\nu(\gamma-1)\kappa]Z\}}{(V-1)[V(V-1)(V-\mu)+(\kappa-\nu V)Z]}, \tag{125}$$

and the eqs. (122) as:

$$\frac{d\ln\zeta}{dV} = \frac{Z-(V-1)^2}{V(V-1)(V-\mu)+(\kappa-\nu V)Z}, \tag{126}$$

and

$$(V-1)\frac{d\ln G}{d\ln\zeta} = (\omega-\nu)V - \frac{V(V-1)(V-\mu)+(\kappa-\nu V)Z}{Z-(V-1)^2}, \tag{127}$$

†The present definition of V and Z differs by factors δ and δ^2 respectively from that employed by Sedov (1959) and Zel'dovich and Raizer (1967).

in which

$$\kappa \equiv \frac{2(\mu-1)+\omega}{\gamma},\ \mu \equiv \frac{1}{\delta},\ \nu \equiv n+1. \tag{128}$$

A large number of self similar gasdynamic flows have been studied. In the book of Sedov (1959) many problems including the strong explosion, the propagation of detonation and combustion fronts, the spherical piston, uniform explosions and implosions, etc. are studied in detail. The classical solution of the converging shock wave (Guderley, 1942), that is a case of self similarity of the second kind, is analyzed in the book of Zel'dovich and Raizer (1967). The recent reviews of Meyer-ter-Vehn and Schalk (1982) on self similar implosions and of Ostriker and McKee (1988) on astrophysical blast waves contain many additional references. Other solutions are scattered in the literature, and we shall not attempt to give a complete list of references. Among them we can mention the self similar implosion of a sphere of gas (Ferro Fontán *et al.*, 1975) and the self similar implosions of gaseous shells (Ferro Fontán *et al.*, 1977). In addition to explosions and implosions, of which some examples will be discussed later one, an important class of self similar motions comprises expansions (see for example Anasimov, 1970, Barrero and Sanmartín, 1977, Sanmartín and Barrero, 1978a, 1978b, Caporaso, 1982, Pakula and sigel, 1985, Schmalz, 1986, Liberman and Velikovich, 1989). Applications to cosmological problems have been discussed by Henriksen and Wesson (1978), Sanyal *et al.* (1985) and Bertschinger (1985a, 1985b). It must be recognized that the family of the self similar solutions of the dynamics of gases is very rich, and although a large number of papers have been published on this subject, there still remain problems to investigate.

5.3 Solutions with Discontinuities

One of the characteristics of gaseous flows is that fronts (shock waves, combustion and detonation fronts, etc.) may occur. At these fronts the density, velocity, temperature, and other variables or parameters of the gas may present discontinuities, or jumps. A self similar solution with a discontinuity is represented in the phase plane by two disjoint pieces of integral curves. The solution passes from one to the other by means of a sudden transition that occurs at a certain fixed value ζ_s of the self similar variable.

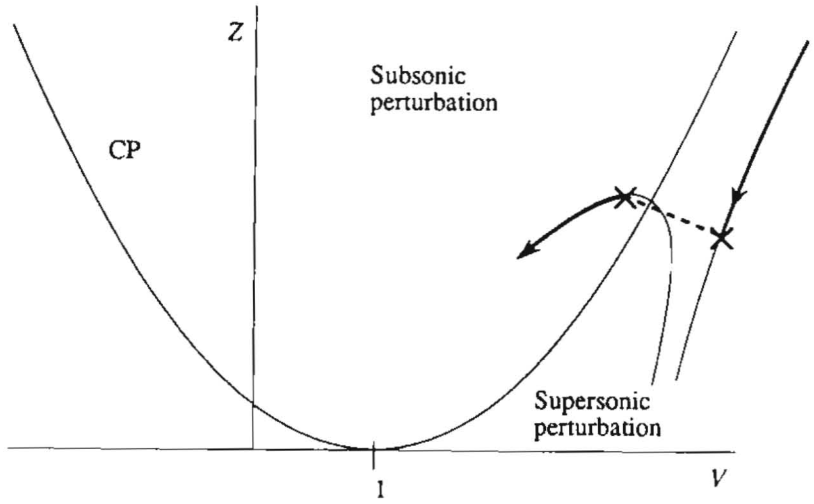

FIGURE 19 Solution with a discontinuity (shock wave).

It can be seen from (121) that $\Delta = 0$ on the parabola $Z = (V - 1)^2$, that is called the critical, or sonic parabola (here we shall use the abbreviation CP). If at a certain point of an integral curve $\Delta = 0$, but $\Delta_1 \neq 0$, $\zeta(V)$ has an extreme. Therefore, if an integral curve crosses the CP at a regular point, in the neighborhood of the point of crossing V is a multivalued function of ζ. But for any physically meaningful solution Z and V must be single valued functions of ζ, so that no part of an integral curve that represents it can cross the CP and continue on the other side along the same curve (the exceptions are those special integral curves that cross the CP through a singular point in which $\Delta_1 = 0$). This means that when the solution we are seeking is represented by a piece of an integral curve that crosses the CP at a regular point, at some place on the curve (no farther than the point of crossing) there must be a discontinuity of the solution (a shock front, or a front of some other nature), so that the rest of it is represented by pieces of other integral curves (see Figure 19).

It is easy to verify that the points of the CP correspond to perturbations that propagate with the (local value of the) speed of sound with respect to the gas. The points of the phase plane below the CP correspond to supersonic perturbations, and those above to subsonic flow. A

shock front will be represented by a discontinuous transition from a point below the CP (that represents the supersonic flow in front of the shock) to a point above it (representing the subsonic flow behind). A detonation front that satisfies the Chapman–Jouguet condition is represented by a transition to a point on the CP (that represents the sonic flow behind the front). The reader can consult the monograph of Sedov (1959), in which the formulae that connect the phase variables on both sides of discontinuities of various kinds (i.e., the Rankine–Hugoniot relationships, the Chapman–Jouguet conditions, etc.) are derived. These formulae must be used whenever appropriate to match the pieces of the integral curves that represent the solution of the problem under study.

For later use we shall give formulae for ordinary shocks. Let us denote by the subscript 1 the quantities on one side of the discontinuity, and by the subscript 2 those on the other side. At a compression shock the conditions of conservation of mass, momentum, and energy must be satisfied. If a shock is moving with a velocity c_s in a perfect gas we can write

$$\rho_1(u_1 - c_s) = \rho_2(u_2 - c_s), \tag{129}$$

$$\rho_1(u_1 - c_s)^2 + p_1 = \rho_2(u_2 - c_s)^2 + p_2, \tag{130}$$

$$\tfrac{1}{2}(u_1 - c_s)^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \tfrac{1}{2}(u_2 - c_s)^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}. \tag{131}$$

These conditions can be written in terms of V, G, and Z by using (117) and by noticing that for self similar flows the shock velocity c_s is given by

$$c_s = \frac{dx}{dt} = \delta\frac{r}{t}. \tag{132}$$

With these substitutions one finds:

$$V_2 = 1 + (V_1 - 1)\left[1 + \frac{2}{\gamma + 1}\frac{Z_1 - (V_1 - 1)^2}{(V_1 - 1)^2}\right], \tag{133}$$

$$Z_2 = \left(\frac{\gamma - 1}{\gamma + 1}\right)^2 \frac{1}{(V_1 - 1)^2}\left[(V_1 - 1)^2 + \frac{2Z_1}{\gamma - 1}\right] \tag{134}$$

$$\left[\frac{2\gamma}{\gamma - 1}(V_1 - 1)^2 - Z_1\right],$$

and

$$G_2 = G_1 \frac{V_1 - 1}{V_2 - 1}. \tag{135}$$

Given the point (V_1, Z_1) in the phase plane that represents the conditions of the gas on one side of the shock one can find from (133), (134) the point (V_2, Z_2) representing the state of the gas on the other side. From the symmetry of (129)–(131) it is clear that the subscripts 1, 2 in (133), (134) can be interchanged. It can be observed that the points of the sonic parabola are transformed into themselves by (133), (134), and also that points above this parabola are transformed into points below it and conversely.

If a strong shock is propagated into a gas at rest, $V_1 = 0$, $Z_1 = 0$ and one obtains

$$V_2 = \frac{2}{\gamma + 1}, \quad Z_2 = \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2}, \quad G_2 = G_1 \frac{\gamma + 1}{\gamma - 1}. \tag{136}$$

These expressions shall be used later on when discussing the strong explosion and the imploding shock wave.

5.4 Particle Trajectories and Characteristics

It is convenient to have at hand the equations that describe the trajectories T of gas elements and of the characteristics, as they help to obtain the physical interpretation of the integral curves representing self similar solutions. The equation of motion of a gas particle is obviously

$$\frac{dx}{dt} = u. \tag{137}$$

Introducing the similarity variables by (117) one obtains:

$$\left(\frac{d \ln \zeta}{d \ln t}\right)_T = \delta(V - 1), \tag{138}$$

in which $\zeta(t)$ denotes the self similar coordinate of the particle. From (138) it follows immediately that $V = 1$ is the condition that $\zeta =$ const. on the trajectory. In consequence, the self similar motion of a free surface is described by $V = 1$. In particular, the self similar motion of a gas-vacuum interface (where $p = 0$) is represented in the phase plane by the singular point $C(V_C = 1, Z_C = 0)$.

Similarly, the equations of the C_+ and C_- characteristics are:

$$\left(\frac{dx}{dt}\right)_{C_\pm} = u \pm c, \tag{139}$$

in which c denotes the local value of the sound velocity. Using (117) one obtains:

$$\left(\frac{d \ln \zeta}{d \ln t}\right)_{C_\pm} = \delta(V - 1 \pm \sqrt{Z}). \tag{140}$$

From this it follows that points on the CP correspond to $\zeta =$ const. characteristics (for $V < 1$ it will be the C_- characteristic, for $V > 1$ the C_+). These limiting characteristics play an important role since they separate flow regions that are in causal contact with the gas at $x = 0$, $t = 0$ from regions that are not causally connected with it, as we shall see below when discussing examples. For a discussion of the limiting characteristics and their role in self similar problems see Witham (1974).

5.5 Unsteady Planar Flows

To acquire familiarity with the application of phase plane methods we shall briefly discuss the self similar planar flows ($n = 0$) in the case $\delta = 1$. This case is instructive and very simple because all the solutions can be obtained in terms of elementary functions. The self similar variable is

$$\zeta = x/bt, \tag{141}$$

in which the constant parameter b has the dimensions of a velocity ($b = c_0$). It must be observed that the solutions we are going to discuss can also be found by the method of characteristics; it is a good exercise for the reader to recover the present results by this route.

When $n = 0$, $\delta = 1$, there is a common factor $Z - (V - 1)^2$ in the numerator and the denominator of (125). For the points that are not on the CP this common factor drops out and we are left with

$$\frac{dZ}{dV} = \frac{2Z}{V}, \tag{142}$$

that can be integrated at once yielding

$$Z = KV^2, \ K = \text{const.}, \tag{143}$$

then from (126) one obtains $\zeta = k/V[k = \text{const.}]$. From (127) it results $G = \text{const.}$. The solutions of (142) are then a family of parabolas with their vertex at the origin of the phase plane. It is easy to see that they represent uniform flows with constant velocity $u = u_0 = Kc_0$; these flows have $z = c_0^2$, and their Mach number is given by $\mathcal{M} = K^{-1/2}$. We notice that in the present case $\Delta = \Delta_1 = 0$ for points of the CP, so that it is allowed to cross the CP along these curves.

It can be verified that the CP itself is also an integral curve; along the CP one has

$$\zeta = k\left|V - \frac{2}{\gamma + 1}\right|^{-1}, \quad G = \left(\frac{V - 1}{V - 2/(\gamma + 1)}\right)^{2/(\gamma - 1)}. \quad (144)$$

This solution represents an expansion, or a compression wave. The density tends to infinity as one approaches the point B, which is located on the CP at

$$V_B = \frac{2}{\gamma + 1}, \quad Z_B = \left(\frac{\gamma - 1}{\gamma + 1}\right)^2. \quad (145)$$

The point B corresponds to $x = \infty$ in the gas. In Figure 20 the phase plane and some integral curves are shown.

Putting together these results, it can be concluded that for $n = 0$, $\delta = 1$, the non trivial self similar solutions will in general consist of fronts, or of expansion and/or compression waves, that connect regions of uniform flow. Let us discuss two examples:

5.5.1 *The expansion of a gas into the vacuum*

The expansion into the vacuum of a gas initially at rest in the $x < 0$ region is represented (see Figure 21) by the following pieces of integral curves: (a) the portion from the origin to the point (0, 1), that represents the gas at rest ($V = 0$) that has not yet been overrun by the front of the expansion wave; (b) the piece of the CP from (0, 1) to infinity, that represents the region $x < 0$ of the expansion wave; (c) the part of the CP from infinity to the vertex at (1, 0) that represents the expansion wave in the region between $x = 0$ and the vacuum, and finally, (d) the segment from (1, 0) to the origin, that represents the vacuum (since $Z = 0$). In this connection it is interesting to comment that if the expanding gas occupied initially a finite volume (for example, if it was contained between two walls at $x = 0$ and at $x = L$, and the last is

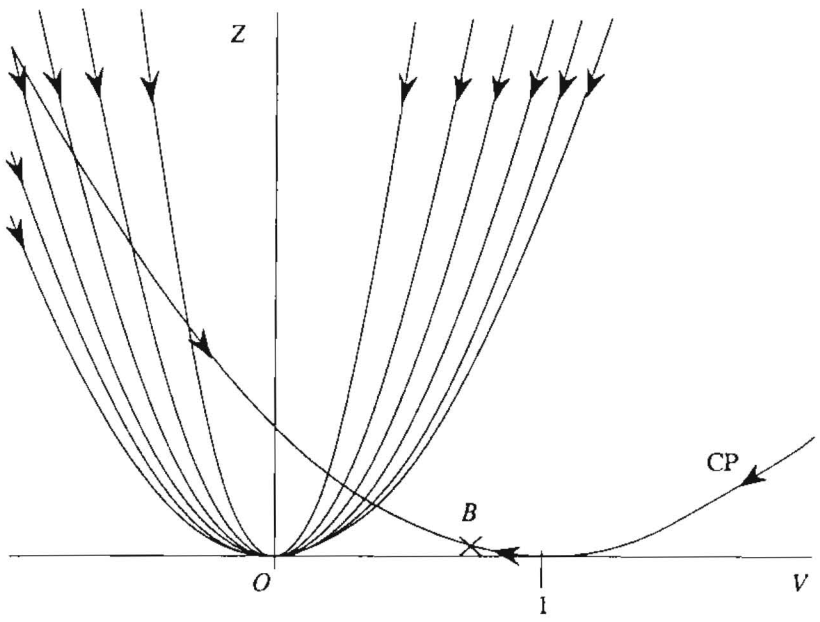

FIGURE 20 Integral curves for $n = 0$, $\delta = 1$; the arrows indicate increasing $|\zeta|$.

removed at the initial moment) the ensuing flow is not self similar (the solution of this problem is discussed in Stanyukovich, 1960). This is one of the cases we mentioned in Section 4.3, in which there is no self similar asymptotics, however large is x/L.

5.5.2 *The centered compression wave*

As a second example consider the flow represented for $t < 0$ by the portion from $(0, \infty)$ to (0, 1) of the $V = 0$ axis, the piece of the CP between (0, 1) and M, and the segment of a parabola (143) between M and the origin of the phase plane (see Figure 22). This self similar solution represents a centered compression wave, all whose C_- characteristics converge at the point $x = 0$ at the time $t = 0$, so that no shock is formed before the compression wave collapses at $x = 0$ (we shall assume that there is a rigid wall at $x = 0$, and that the gas occupies the region $x > 0$). Then, the first piece represents the gas at rest in front of the compression wave; the piece of the CP describes the compression

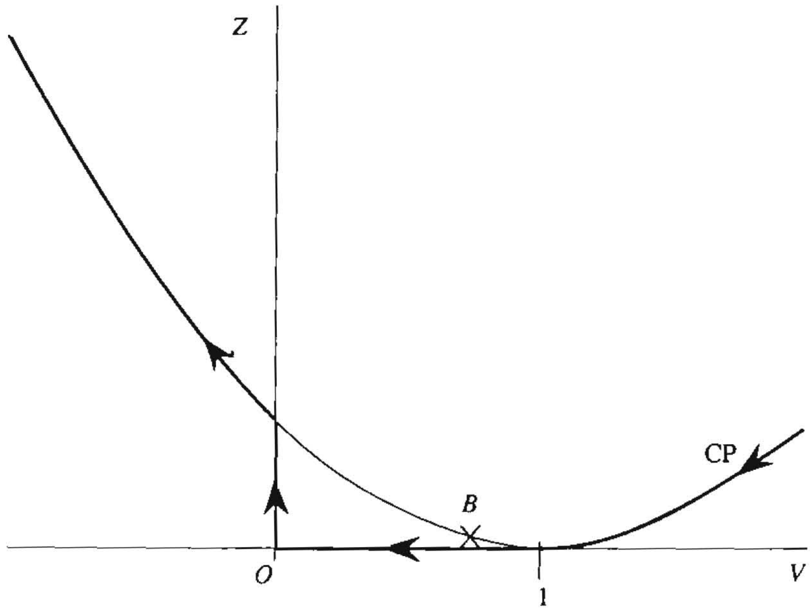

FIGURE 21 Integral curves representing the expansion of a gas into vacuum; the arrows indicate increasing ζ.

wave, and the third piece describes the dense gas, behind the compression wave, as it moves towards $x = 0$. At the precise instant of collapse, $t = 0$, the gas has everywhere the same density (that of the dense gas) and moves with constant velocity.

To continue the solution from $t < 0$ to $t > 0$, i.e., to find the flow after the collapse, one must keep in mind that according to (117) the sign of V must change when the sign of t changes. Hence the integral curve representing the solution for $t > 0$ must lie in the $V < 0$ half plane, and must have the same Mach number as that for $t < 0$.† Then the portion from $(0,\infty)$ to $(0,RS')$ of the $V = 0$ axis, and the piece of parabola from RS to the origin in Figure 22 describe what happens after the collapse: the gas that is still converging towards the origin (second piece) encounters an outgoing shock wave (the jump

†This is necessary to ensure continuity of the physical variables.

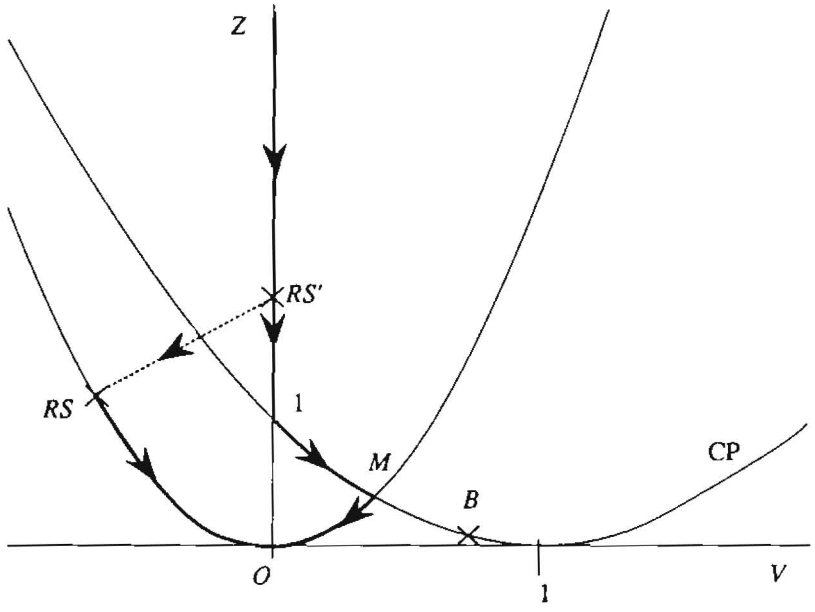

FIGURE 22 Integral curves representing the planar self similar implosion; the arrows indicate increasing ζ.

$RS - RS'$) and is further compressed and brought at rest in the region near the wall.

The closer is M to B, i.e., the larger the Mach number of the uniformly converging gas behind the compression wave, the stronger will be the compression. Notice that an infinite compression is achieved for a finite $\mathscr{M} = \mathscr{M}^* = 2/(\gamma - 1)$. It is not possible to have centered compression waves with $\mathscr{M} > \mathscr{M}^*$. A self similar compression wave with $\mathscr{M} > \mathscr{M}^*$ must be represented in the phase plane by a piece of the CP going from the singular point C to M (which now lies to the right of B), which means that there will be an empty cavity extending from the origin to the front of the compression wave (the point C represents a boundary between the gas and a vacuum).

The self similar solution of Figure 22 corresponds for planar symmetry to the self similar spherical implosion studied by Ferro Fontán *et al.* (1975). In the present case the solutions are fully analytic; they can be derived by the method of characteristics, considering the flow generated by a piston that advances towards $x = 0$, and imposing the condition that the compression wave is centered (in other words, that the motion of the piston is such that all the C_- characteristics emanating from it collapse at the point $x = 0$ when $t = 0$). For spherical or cylindrical implosions the relevant integral curves are not analytical and must be calculated numerically. An interesting analogy to the plane centered compression wave we have been discussing has been given by Lengyel (1973).

5.6 Strong Explosions

Explosions are very important for astrophysicists, and the interest in explosive phenomena is due not only to scientific but also to practical reasons. Apart from the nature or origin of the explosive process itself, the expanding blast wave will shock, heat, and accelerate the surrounding ambient medium. If there are many explosions, the various interacting blast waves may dominate over other physical processes to the extent that they determine the overall properties of the medium. A very extensive and recent review paper on astrophysical blast waves is due to Ostriker and McKee (1988) and contains a large list of references. In this section we shall briefly discuss the classical Sedov–Taylor (Sedov, 1946, 1959; Taylor, 1950) solution to the problem of a strong explosion, which is the prototype of these type of problems.

Let us consider that at $t = 0$ an explosion occurs at the center of symmetry ($x = 0$) of a gas at rest, in which a finite amount of energy E_0 is liberated instantaneously (for the analogous problems with cylindrical and plane symmetry E_0 will be the energy per unit length, or per unit area, respectively). We shall be primarily interested in the case in which the gas has initially a uniform density ρ_0 (i.e., $\omega = 0$ in (117)), although most of the formulae that we shall present are also valid if the initial density varies according to a power law. We shall neglect the mass and dimensions of the object that liberates the energy. This means that the very first stages of the phenomenon will not be adequately described by the theory, that applies only after the blast wave has extended to the point when the swept-up and shocked mass of gas greatly exceeds that of the original object. We shall also neglect the radiative transfer of energy from the explosion region to its surroundings, which is the dominant mechanism at the beginning: during this

initial stage the coefficient of thermal conductivity λ can be considered a power function of the temperature ($\lambda = \lambda_0 T^m$, in which λ_0 is a constant, and $m \approx 5$, see Section 7.1); in this phase of the phenomenon a strong nonlinear thermal wave whose front expands according to the law $x_f \sim t^{1/(2+\nu m)}$ heats the gas around the point of the explosion (see for example Barenblatt, 1979). An intense shock wave arises in the heated gas which soon outstrips the thermal wave, so that subsequently the phenomenon enters into a purely gasdynamic stage. We shall focus our attention on this second stage, in which the constant dimensional governing parameters will be ρ_0 and p_0, the initial density and pressure of the gas, and E_0. The motion of the gas will depend, under adiabatic conditions, on the following parameters:

$$\rho_0,\ p_0,\ E_0,\ x,\ t,\ \gamma. \tag{146}$$

By considerations of dimensional analysis, it is then found that all the independent dimensionless quantities of the problem can only depend on the following dimensionless parameters:

$$\zeta = \frac{x}{(E/\rho_0)^{1/(2+\nu-\omega)}\, t^{2/(2+\nu-\omega)}},\quad \tau = \frac{p_0^{(2+\nu-\omega)/2\nu}\, t}{E^{1/\nu}\rho_0^{1/2}},\quad \gamma, \tag{147}$$

where $E = \alpha E_0$ is a constant parameter whose dimensions are those of energy ($\nu = 3$), or energy per unit length ($\nu = 2$), or energy per unit area ($\nu = 1$), and α is a numerical constant whose value will be determined later. In (147), ζ and τ are variables so that the flow is not in general self similar. Nevertheless, both experiment and theory show that at the boundary of the region of disturbed gas motion during an explosion a shock wave is formed. In a spherically symmetric explosion ($\nu = 3$) the shock will be a sphere whose radius increases with time. Clearly, the influence of the initial pressure, and then of τ, enters only due to the shock conditions. But then, in the limit of a very strong explosion (i.e. if E_0 is large) the pressure behind the shock wave will be much larger than p_0, and the gas motion behind the shock wave will be practically independent of p_0. This will happen for t not too large, so that the radius of the shock front is still small and $\tau \ll 1$. In this situation only two constant dimensional parameters govern the problem, ρ_0 and E_0, and the motion is self similar. Notice that for large t, as the shock wave attenuates further, it is not correct to neglect the counterpressure p_0, so that the gas motion ceases to be self similar.

Summarizing the previous discussion, the self similar regime in which we are interested occurs for t not too small nor too large, such that

$$t_M,\ t_R \ll t \ll t_P, \tag{148}$$

where

$$t_M \approx \frac{1}{\sqrt{\varepsilon}}\left(\frac{M}{\rho_0}\right)^{\frac{1}{\nu}},\quad t_R \approx \left[\left(\frac{E_0}{\rho_0}\right)^{2(m-1)}\left(\frac{\kappa}{C^m}\right)^{2+\nu}\right]^{\frac{1}{2+\nu(2m-1)}}, \tag{149}$$

$$t_P \approx \left(\frac{E_0}{p_0}\right)^{\frac{1}{\nu}}\left(\frac{\rho_0}{p_0}\right)^{\frac{1}{2}}.$$

In (132) t_M denotes the time when the shock has swept a mass of the order of M, the mass of the exploding object (then for $t \gg t_M$ the latter can be neglected), t_R denotes the time when the shock overtakes the radiative thermal wave, and t_P is the time when the pressure behind the shock becomes comparable to p_0. In (149) ε denotes the specific energy of the explosion (energy liberated per unit mass of the explosive), C is the specific heat (per unit mass) of the gas ($\nu = 3$), or the specific energy per unit length or area ($\nu = 2, 1$, respectively), and $\kappa = \lambda_0/(m + 1)C$. Clearly, $\tau = t/t_P$.

The hydrodynamic self similar regime will exist provided

$$t_M,\ t_R <<< t_P, \tag{150}$$

in the sense discussed in Section 4. We are here in the presence of a case of self similarity of the first kind, as the similarity exponent

$$\delta = 2/(2 + \nu - \omega), \tag{151}$$

is determined by dimensional analysis alone.

The solution in which we are interested will be represented in the phase plane by an integral curve, solution of (121), beginning at the point S, whose coordinates are given by (136):

$$V_S = \frac{2}{\gamma + 1},\quad Z_S = \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2}, \tag{152}$$

that describes the state of the gas immediately behind the strong shock that is advancing in the gas at rest (see Section 5.3). It results that this integral curve is analytic, being given by the following formula:

$$Z = -\frac{V^2(V - 1)(\gamma - 1)}{2(V - 1/\gamma)}, \tag{153}$$

as can be easily verified.

Three singular points of (121) that lie on the curve (153) are relevant for the present problem:

(a) The point $D(V_D = 1/\gamma,\ Z_D = \infty)$ is a saddle point; there is a single integral curve arriving at D from the finite (V, Z) plane, which is given by (153). Along this curve $\zeta \to 0$ as D is approached, so that this point represents the origin of coordinates in the gas.

(b) The point $C(V_C = 1,\ Z_C = 0)$ is a node, it represents a point at a finite distance from the origin (for finite t), where the density and the pressure of the gas vanish.

(c) The point E

$$V_E = \frac{2}{\delta[2 + (\gamma - 1)\nu]}, \tag{154}$$

$$Z_E = -\frac{2\gamma(\gamma - 1)[(2 - \gamma)\nu - \omega]}{\delta^2[(2 - \omega)\gamma + \nu - 2][2 + (\gamma - 1)\nu]^2},$$

is a node; as E is approached along the curve (153), $\zeta \to \infty$, so that this point represents the state of the gas at infinity.

As ω or γ are varied the point E moves along the integral curve (153), and can pass through C from the $Z < 0$ half plane to the upper $Z > 0$ half plane; the condition that Z_E be positive is given by

$$(2 - \gamma)\nu \leqslant \omega \leqslant \frac{2(\gamma - 1) + \nu}{\gamma}. \tag{155}$$

Furthermore, E can pass through S, so that its position may be intermediate between D and P. This happens in the intervals

$$\frac{2(\gamma - 1) + \nu(3 - \gamma)}{\gamma + 1} \leqslant \omega \leqslant \frac{2(\gamma - 1) + \nu}{\gamma} \tag{156}$$

that are represented as hatched areas in Figure 23a–c for $\nu = 1, 2, 3$. It can be appreciated that for an explosion in a gas of uniform density ($\omega = 0$) this situation happens only for spherical symmetry, for $\gamma > 7$.

There is an important difference in the nature of the solution according if E is located between D and S, or lies outside this interval:

(i) Let us discuss the last situation, that occurs for $\omega < [2(\gamma - 1) + \nu(3 - \gamma)]/(\gamma + 1)$, or $\omega > [2(\gamma - 1) + \nu]/\gamma$ (unshaded areas in Figure 23). In this case the solution is represented by

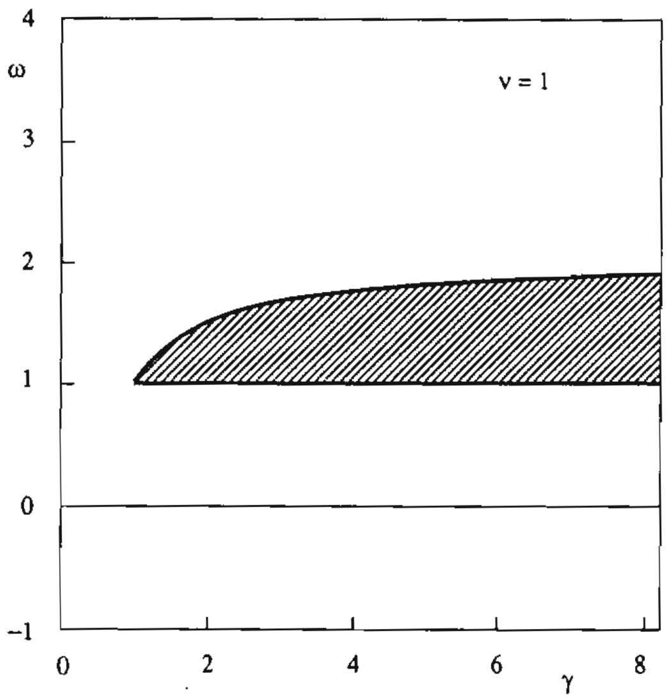

FIGURE 23a Location of the singular point $E(\nu = 1)$: in the shaded intervals E lies between D and P.

the piece of the integral curve (153) that joins D with S, which describes the motion of the gas from the origin to the shock front. This is shown in Figure 24 for $\nu = 3$, $\gamma = 5/3$, $\omega = 0$. As can be observed in Figure 23, if $\omega = 0$ and $\nu = 1, 2$, E lies outside the interval $D - S$ for any $\gamma > 0$; for $\nu = 3$ this is true only for $\gamma < 7$.

It is easy to find expressions of the physical variables in terms of V; we give here the formulae for the case $\omega = 0$:

$$\zeta = \zeta_S \left(\frac{V}{V_S}\right)^{\alpha_{0\zeta}} \left(\frac{V - V_D}{V_S - V_D}\right)^{\alpha_{D\zeta}} \left(\frac{V - V_E}{V_S - V_E}\right)^{\alpha_{E\zeta}}, \tag{157}$$

$$\rho = \rho_S \left(\frac{V - 1}{V_S - 1}\right)^{\alpha_{1\rho}} \left(\frac{V - V_D}{V_S - V_D}\right)^{\alpha_{D\rho}} \left(\frac{V - V_E}{V_S - V_E}\right)^{\alpha_{E\rho}}. \tag{158}$$

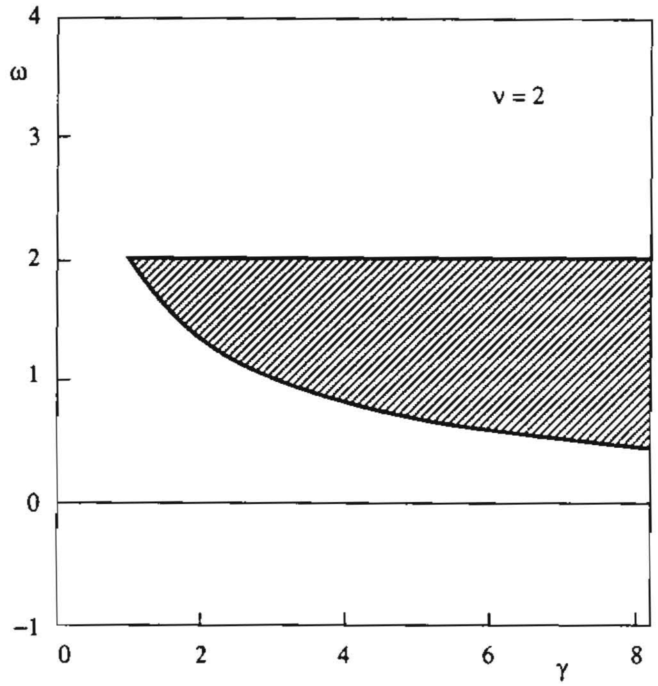

FIGURE 23b Location of the singular point $E(\nu = 2)$: in the shaded intervals E lies between D and P.

$$p = p_S \frac{(1 + \gamma)^2 V_S^2}{4\gamma} \frac{V_S - 1}{V_S - V_D} \left(\frac{V}{V_S}\right)^{\alpha_{0p}} \left(\frac{V - 1}{V_S - 1}\right)^{\alpha_{1p}} \left(\frac{V - V_E}{V_S - V_E}\right)^{\alpha_{Ep}}, \tag{159}$$

$$T = T_S \frac{(1 + \gamma)^2}{4\gamma} \frac{V_S^2(V - 1)}{(V_S - V_D)^2} \left(\frac{V}{V_S}\right)^{\alpha_{0p}} \left(\frac{V - V_D}{V_S - V_D}\right)^{-\alpha_{Dp}} \left(\frac{V - V_E}{V_S - V_E}\right)^{2\alpha_{E\zeta}} \tag{160}$$

with

$$\alpha_{0\zeta} = -\frac{2}{2 + \nu}, \quad \alpha_{D\zeta} = \frac{\gamma - 1}{2(\gamma - 1) + \nu},$$

$$\alpha_{E\zeta} = \frac{1}{2 + \nu(\gamma - 1)} \left[\frac{2\nu(\gamma - 2)}{2 + \nu} - \gamma(2 + \nu)\alpha_{D\zeta}\right],$$

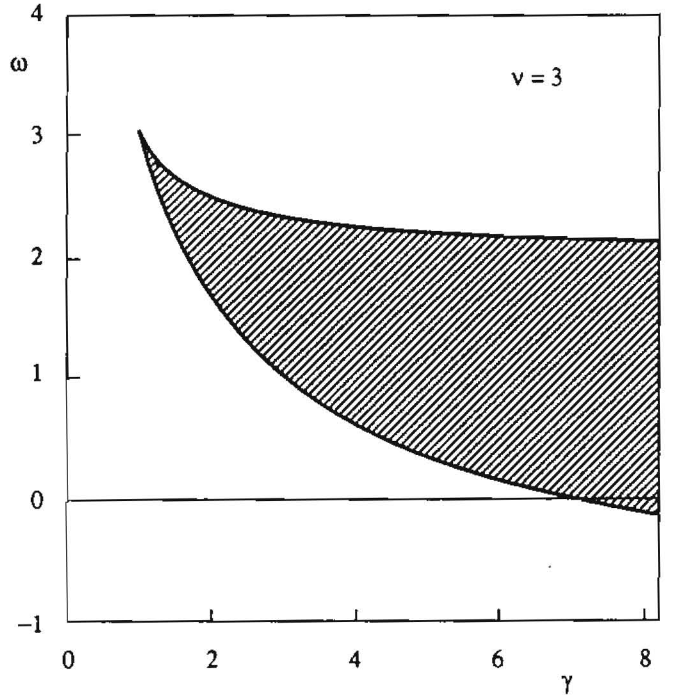

FIGURE 23c Location of the singular point $E(\nu = 3)$: in the shaded intervals E lies between D and P.

$$\alpha_{1\rho} = \frac{2}{\gamma - 2}, \quad \alpha_{D\rho} = \frac{\nu}{2(\gamma - 1) + \nu}, \quad \alpha_{E\rho} = \alpha_{E\zeta}\frac{2 + \nu}{\gamma - 2}, \tag{161}$$

$$\alpha_{0p} = \frac{2\nu}{2 + \nu}, \quad \alpha_{1p} = \frac{\gamma}{\gamma - 2}, \quad \alpha_{Ep} = \alpha_{E\zeta}\frac{2(\gamma - 1) + \nu}{\gamma - 2}.$$

The dependencies of the physical variables on ζ are represented in Figure 25 for the case $\nu = 3$, $\gamma = 5/3$, $\omega = 0$; results for other values of ν and γ are given in the book of Sedov (1959).

It can be appreciated that the velocity is zero at the center of symmetry, and increases almost linearly with the coordinate near to it. The pressure is finite, and tends to a constant value as the origin is approached. The density tends to zero very rapidly as $\zeta \to 0$, so that most

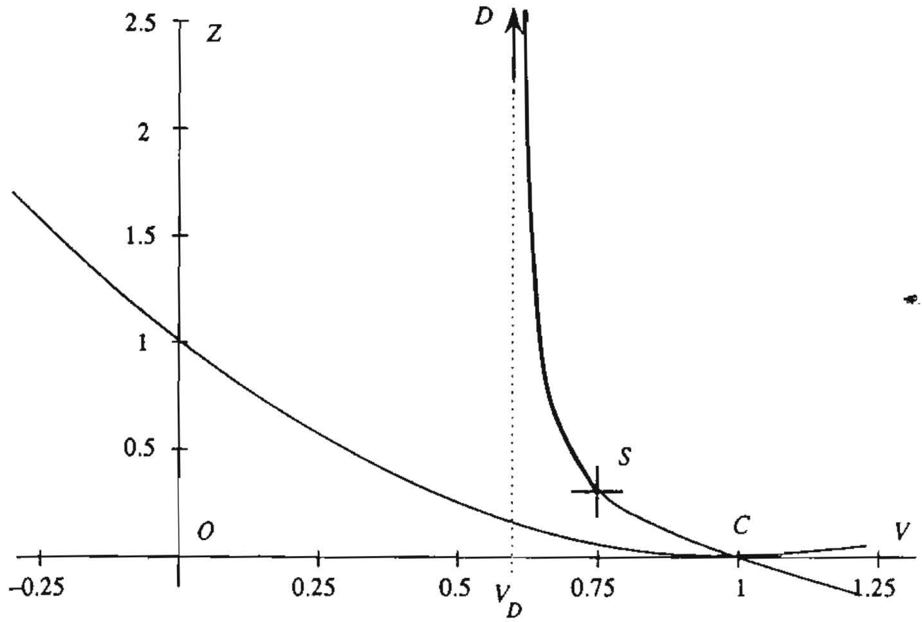

FIGURE 24 Integral curve for a strong explosion ($\nu = 3$, $\gamma = 5/3$, $\omega = 0$).

of the disturbed gas is contained in a rather thin shell just behind the shock front. The temperature tends to infinity, hence very large temperature gradients occur near the center of the explosion. In this situation it is to be expected that heat conduction (that we neglected) will be very important; if this is taken into account it can be shown that T is finite at $\zeta = 0$ (see Sedov, 1959).

(ii) Now let us consider the first situation, i.e. $[2(\gamma - 1) + \nu(3 - \gamma)]/(\gamma + 1) < \omega < [2(\gamma - 1) + \nu]/\gamma$ (hatched intervals in Figure 23); for an explosion in a uniform gas ($\omega = 0$) this will occur only in the case of spherical symmetry ($\nu = 3$), and for $\gamma > 7$. When E is located between D and S it is not possible to find a solution extending from the origin of coordinates to the shock, because as one moves along the curve (153) starting from D, ζ increases to infinity as E is approached before arriving to the point S. Clearly the piece DE of the integral curve cannot represent the solution of the problem we are considering, which must now be represented by another piece of the

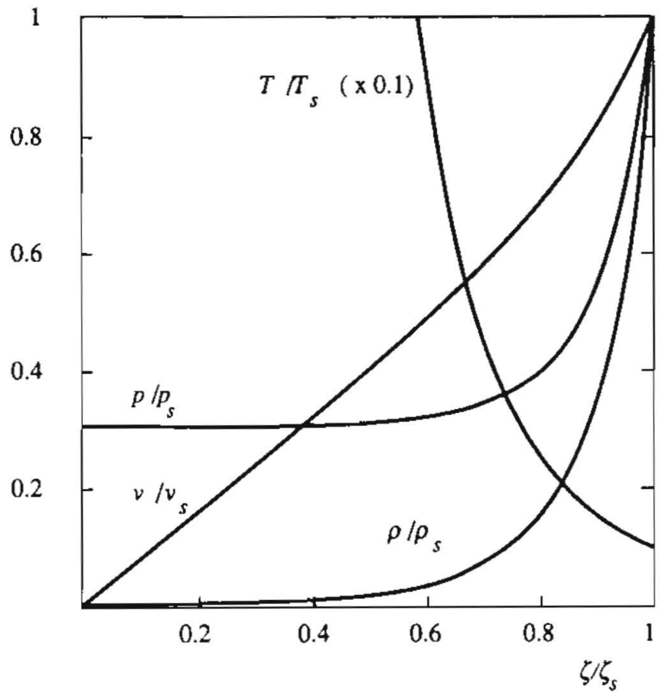

FIGURE 25 Profiles of the physical variables for a strong explosion ($\nu = 3, \gamma = 5/3, \omega = 0$).

curve (153). The piece we need can be no other than that which joins C with S. This is shown in Figure 26, that corresponds to $\nu = 3$, $\gamma = 11, \omega = 0$. In the following we shall consider only $\nu = 3, \omega = 0$.

As stated above, C represents a moving point of the gas, at a finite distance from the origin of coordinates. Its position is given by

$$x_C = \zeta_C(E/\rho_0)^{1/5}t^{2/5}, \tag{162}$$

with

$$\zeta_C = \zeta_S 4^{1/5}(1+\gamma)^{-\frac{\gamma+1}{3\gamma-1}}\left(\frac{\gamma-7}{3(\gamma-2)}\right)^{\frac{25\gamma(\gamma-1)-6(\gamma-2)(2\gamma+1)}{5(3\gamma-1)(2\gamma+1)}}. \tag{163}$$

At C, the density, the pressure and the temperature of the gas vanish, so that an empty cavity whose radius is x_C is produced by the explosion.

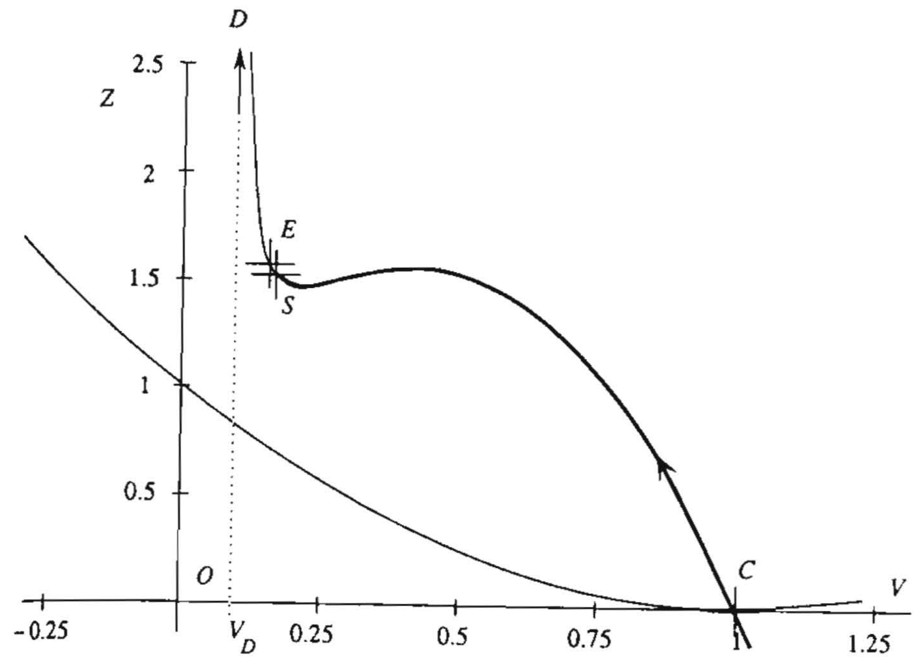

FIGURE 26 Integral curve for a strong explosion in which a central cavity is formed ($\nu = 3, \gamma = 11, \omega = 0$).

On the other hand, the velocity tends to infinity as C is approached. The piece CS of (153) describes the motion of the gas between x_C and x_S.

The dependency of ζ_C with γ is shown in Figure 27. It can be observed that ζ_C/ζ_S is always less than $\approx 1/4$, and becomes vanishingly small as $\gamma \to \infty$.

The profiles of the physical variables are given as before by (157)–(161), and are represented in Figure 28 for the case $\gamma = 11$.

To complete the solution of the problem of the strong explosion it remains to determine the value of the numerical constant α that fixes the coordinate scale in (147). For this purpose one must calculate the total energy of the explosion, that is given by:

$$E_0 = \alpha(\gamma)E = \sigma_\nu \int_0^{x_S}\left[\tfrac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right]x^{\nu-1}dx, \tag{164}$$

$$\sigma_1 = 2,\ \sigma_2 = 2\pi,\ \sigma_3 = 4\pi,$$

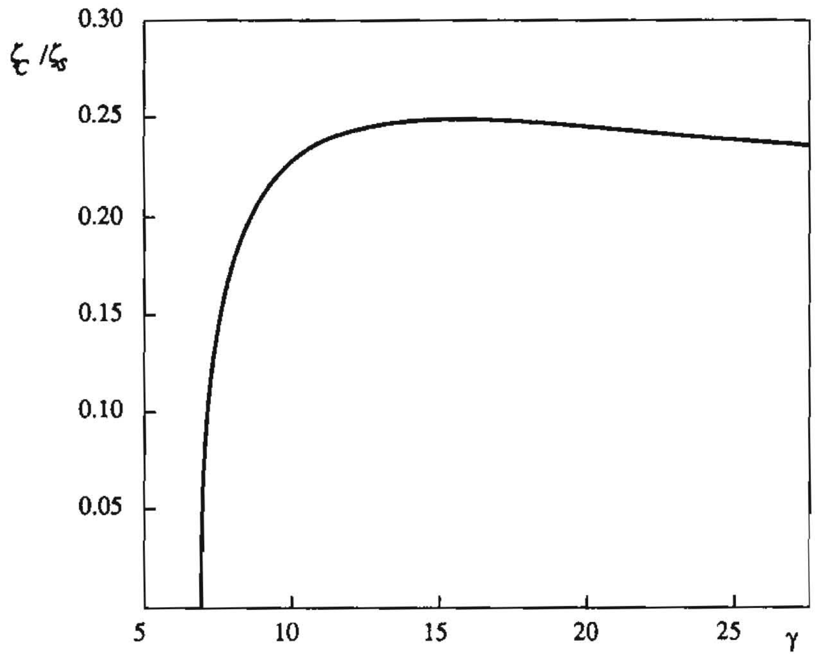

FIGURE 27 Radius of the cavity produced by a strong explosion for $\gamma > 7$ ($\nu = 3$, $\omega = 0$).

The first term in the integral is the kinetic energy density and the second term is the internal energy of the gas. In terms of the dimensionless functions (117) one obtains, using (153):

$$\alpha(\gamma) = \frac{2\sigma_\nu(\gamma + 1)}{(\nu + 2)^2} \int_{0,\zeta_C}^{\zeta_s} \frac{GV^3}{\gamma V - 1} \zeta^{\nu+1} d\zeta, \tag{165}$$

in which the lower limit of the integral is 0 if the solution is represented by the piece DS of (153) and ζ_C if by CS. The integration can be effected by changing to the variable V and using (157)–(161). The function α is shown in Figure 29.

Having thus fixed the coordinate scale we can give expressions of the position and velocity of the shock front. For $\omega = 0$ one has:

$$x_S = (E_0/\alpha\rho_0)^{1/(2+\nu)} t^{2/(2+\nu)}, \quad u_S = \frac{2}{2+\nu} (E_0/\alpha\rho_0)^{1/(2+\nu)} t^{-\nu/(2+\nu)} \tag{166}$$

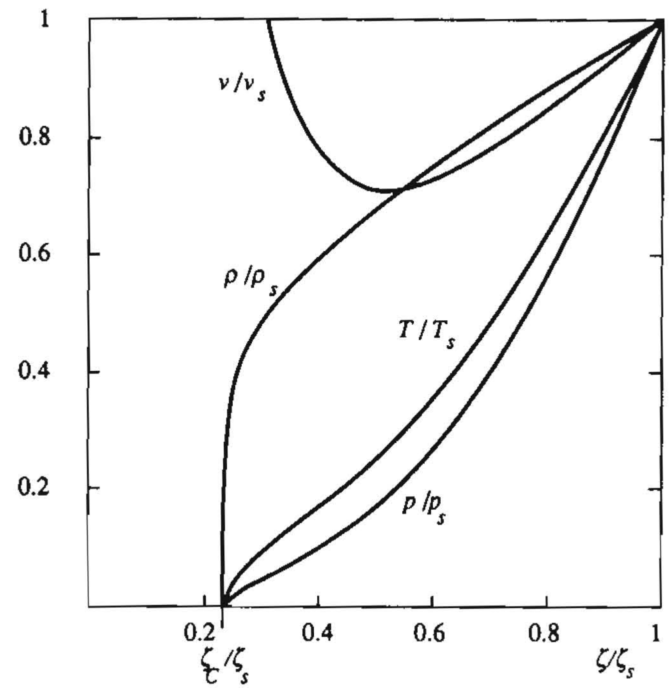

FIGURE 28 Profiles of the physical variables for a strong explosion in which a central cavity is formed ($\nu = 3$, $\gamma = 11$, $\omega = 0$).

Other problems related to the strong explosion we have been considering, as well as discussions on various effects such as heat conduction, counterpressure, variable density of the medium, etc. can be found in the book of Sedov (1959).

5.7 Implosions

Implosions are an important class of phenomena whose effect is to concentrate a large amount of eneryg in a small volume (a process called cumulation) at the instant of culmination; at later times this high energy density drives an outgoing shock wave similar to that of an explosion (however this shock is not as strong as that discussed in the preceding Section, as the counterpressure cannot be neglected as before). Various interesting self similar solutions have been found that

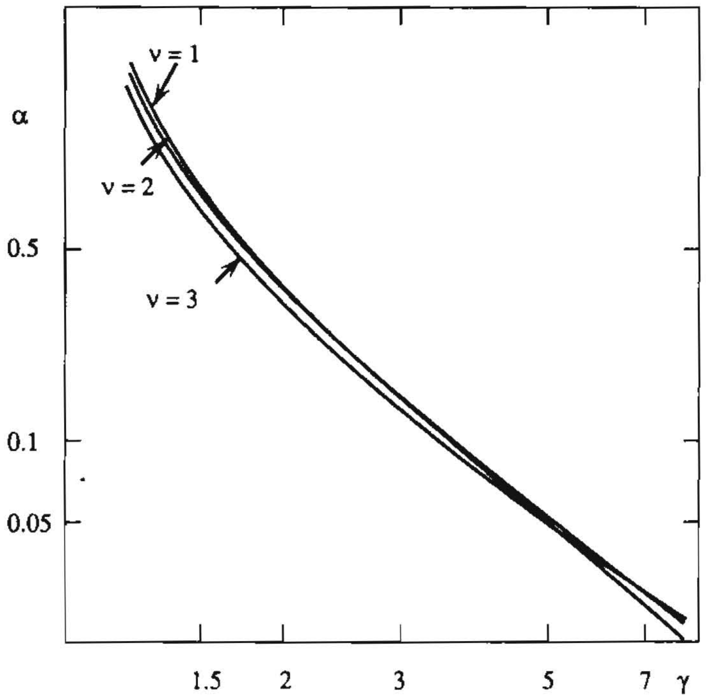

FIGURE 29 The ratio E_0/E as a function of γ; notice the logarithmic scale.

describe implosions. Classical problems of this type are the collapse of a cavity in water (Rayleigh, 1917, Hunter, 1960) and the converging shock wave (Guderley, 1942, Yousaf, 1986). Research in inertial confinement fusion, in which one strives to produce large densities and a high concentration of energy in a central region of a target (a small pellet) to initiate nuclear fusion reactions, has spurred interest in implosions, and many papers have been published describing various types of self similar implosions. A huge amount of work has also been done in the development of sophisticated codes for the numerical simulation of implosions and the design of pellets, and a tremendous effort has been spent in multimillion dollar experiments in which various kinds of targets have been compressed to very high densities by extremely powerful laser beams. Implosions are also relevant for other fusion concepts such as dynamic pinches and liners. I shall not discuss

here these matters as the interested reader can find excellent reviews elsewhere, and more information can be found in the references. A comprehensive review on this matter has been written by Meyer-ter-Vehn and Schalk (1982), where additional references can be found. We shall discuss with some detail the converging shock wave, as it offers a fine example of a self similar solution of the second kind in gas dynamics (see Zel'dovich and Raizer, 1967, also Landau and Lifschitz, 1979).

Consider a spherically (or axially) symmetric flow in which a strong shock wave travels towards the center (or axis) through a medium of uniform initial density ρ_0. We shall not discuss what caused the wave; it could have been produced, for instance, by a spherical (or cylindrical) piston which pushed the gas inward, imparting a certain amount of energy to it. We shall concentrate our interest in what happens in the advanced stages of the phenomenon, when the shock front is arriving at the center (or axis) and immediately after its collapse, that we assume occurs at $t = 0$. We shall also be concerned with the motion of the medium for small values of the radius.

As the shock wave converges, the energy concentrates near the front (cumulation) and the wave strengthens; then the unperturbed pressure p_0 of the gas inside plays no role in determining the motion behind the shock front, and can be taken as zero. It is then reasonable to assume that close to the moment of collapse and near the center (or axis) the motion will approach some limiting regime (that we are going to determine) in which the initial conditions have been "forgotten" to a considerable extent.

In the limiting, or asymptotic regime in which we are interested the problem does not contain characteristic parameters with dimensions either of length or of time. The initial radius of the "piston" cannot be a scale of length in a region very small as compared to it. The only scale of length is the radius x_S of the shock front, that is a function of time; the scale of velocity is the velocity of the front dx_S/dt, that is also dependent on time. Then we expect that the asymptotic regime will be self similar.

Notice that in this case the self similarity exponent δ cannot be determined in advance. In fact, apart from the initial density ρ_0 there are no other parameters that can be used to construct the self similar variable ζ. In this connection it must be observed that the energy imparted to the gas by the piston (that has a definite value) cannot be taken as a parameter, since only a small part of it is concentrated in the self similar region (whose radius is of the order of x_S), and this fraction decreases with time. It is then clear that the solution must be a self similarity of the second kind. The dimensions of the parameter b in (117), related to the similarity exponent δ, are not known in advance. If δ is found by direct construction of the solution, as we shall do, then the numerical value of b will be still indeterminate. It depends on the initial conditions of the problem, and can be obtained only by experiment, or by following the (numerical) solution of the full non degenerate problem until its self similar asymptotics is approached.

As said before, the self similar solution holds only in a small region, whose size is of the order of x_S, and then, only close to $t = 0$, when x_S is small. If one solves the full problem of the motion of the medium, with appropriate initial conditions so that an imploding shock is produced, it will be found that as the moment of the collapse is approached the true solution will approximate closer and closer the asymptotic self similar solution in a region near the shock front. The form of the asymptotic solution does not depend on the initial condition, nor on the motion of the medium at large distances ($x >> x_S$); in particular it is independent on how the motion was originated. However the asymptotics is not entirely independent on the initial conditions, because it selects from all this information a single datum, the numerical value of b. This value characterizes the intensity of the initial push that set the medium in motion.

While the form of the asymptotic solution is independent on the initial conditions and on the motion at large distances, the manner in which the true solution tends to this asymptotics does, on the contrary, depend on it. The closer the initial motion corresponds to the limiting motion, the sooner will the true motion near the front attain the self similar regime. But it will reach it anyway sooner of later, regardless of the initial conditions and of the motion at large distances.

Let us now show how the self similar solution is constructed. We consider first the motion before collapse ($t < 0$). To find the desired integral curve we observe that the point S that represents in the phase plane the motion of the gas just behind the shock ($x = x_S$) is given by

$$V_S = \frac{2}{\gamma + 1}, \quad Z_S = \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2}, \tag{167}$$

according to (136). The integral curve will describe the motion for $x > x_S$, and it must be possible to consider arbitrarily large values of

x, i.e. $\zeta \to \infty$. Then it must begin at S and end at the singular point O ($V_O = 0$, $Z_O = 0$) that represents the infinity in the gas (Sedov, 1959). The point O is a node, through which pass an infinity of curves that in its neighborhood are given by $Z = KV^2$, with $K = \text{const.}$, as in (143); they represent flows that for large x are converging uniformly, being characterized by their asymptotic Mach number $\mathcal{M}_\infty = K^{-1/2}$.

Now we observe that in order to join P and O our integral curve must intersect the CP. The crossing cannot occur through a regular point of (125) because then V, Z would be multivalued functions of ζ, which is physically inacceptable as commented in Section 5.3. Thus the intersection must occur at a singularity of (125). It can be checked that there may be two such singular points on the CP, that we call B_+ and B_- whose coordinates are given by:

$$V_{B\pm} = (p \pm q)/2n, \quad Z_{B\pm} = (V_{B\pm} - 1)^2, \tag{168}$$

in which ($\omega = 0$):

$$p = \kappa + \nu - \mu, \quad q = (p^2 - 4n\kappa)^{1/2}. \tag{169}$$

If one specifies some arbitrary value of δ and integrates (125) starting from S, the resulting integral curve in general either will have no intersection with the CP or will cross it at a regular point; then this curve will not give the correct solution. Only for a special value of δ the curve will cross the parabola passing through the appropriate singular point (that must be either B_+ or B_-), after which it will go to O. The requirement that the desired integral curve must pass through the appropriate singular point determines the similarity exponent δ. Thus the latter is found by solving a nonlinear eigenvalue problem, as is typical of self similarity of the second kind.

The points B_+, B_- are real if $(p^2 - 4n\kappa) > 0$, which happens for $\delta > \delta_+$ or $\delta < \delta_-$ with

$$\delta_\pm = \frac{\gamma + 2 \pm \sqrt{8\gamma}}{\nu\gamma + 2 \pm \sqrt{8\gamma}}. \tag{170}$$

These intervals are represented in Figure 30 for $\nu = 2, 3$. It can be shown that $\delta > \delta_+$ is the interval of interest.

The eigenvalue δ is found by trial and error, integrating numerically (125). Exponents δ have been calculated for various γ (Welsh, 1967, Lazarus and Richtmeyer, 1977, Rodriguez and Linan, 1978, Brushlinski and Kazhadan, 1963). In the present paper, they have been

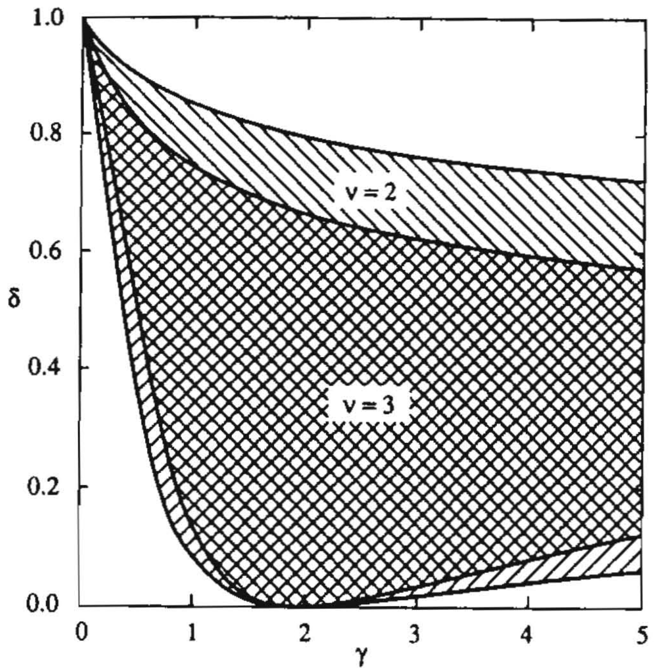

FIGURE 30 In the shaded regions there are no singular points on the CP ($\omega = 0$).

calculated for 20 values of γ in the interval $1 \leqslant \gamma \leqslant 3$.† Figures 31, 32 show the dependency of δ with γ for $\nu = 2, 3$, respectively. It can be observed that δ_+ gives a good approximation to the eigenvalue (Stanyukovich, 1960, Yousaf, 1986, Fujimoto and Mishkin, 1978a,b). It is to be also observed that there is a certain value $\gamma = \gamma_c$ such that $\delta = \delta_+$ (for $\nu = 3$, $1.8 < \gamma_c < 1.9$, see Yousaf, 1986); there, $q = 0$, and B_+, B_- coincide. For $\gamma < \gamma_c$ the integral curve that gives the solution of the problem passes through B_+, that is a saddle point. For

†All the numerical data quoted in this Section as well as those displayed in the graphs have been recalculated. The values of δ represented in Figures 31, 32 are in perfect agreement with those of the literature. Nevertheless it should be noticed that the present calculations yield values of the compression factor of the implosion that differ by a few percents from those quoted by Zel'dovich and Raizer (1967); we shall not discuss this matter in detail since it is beyond the scope of the present review, and will be addressed in a future publication.

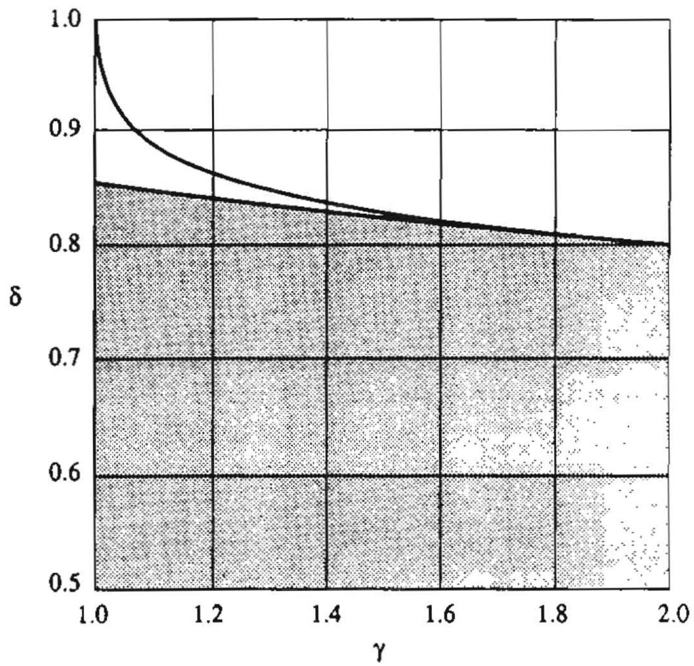

FIGURE 31 The eigenvalue δ as a function of γ for $\nu = 2$. The line that borders the gray region is $\delta = \delta_+$.

$\gamma > \gamma_c$ the appropriate integral curve passes through B_-, that is a node. In either case there is a single trajectory through the singular point on the CP having the required regular properties.

The solution for $t > 0$ that describes the flow after the shock collapses at the center can be constructed (Guderley, 1942) as discussed in Section 4.5 in connection with the centered plane compression wave, by considering that in the neighborhood of O (i.e. for $x \to \infty$) the integral curve representing the solution for $t > 0$ must have the same asymptotic Mach number as that corresponding to $t < 0$.

The integral trajectories are represented in Figure 33 ($\nu = 3$, $\gamma = 7/5$). The solution for $t < 0$ is represented by the curve SB_+O, and that for $t > 0$ by the two pieces DRS_1 and RS_2O. An outgoing shock wave is formed, corresponding to the discontinuous transition from RS_1 to RS_2. This discontinuity is necessary in order to extend the solution to

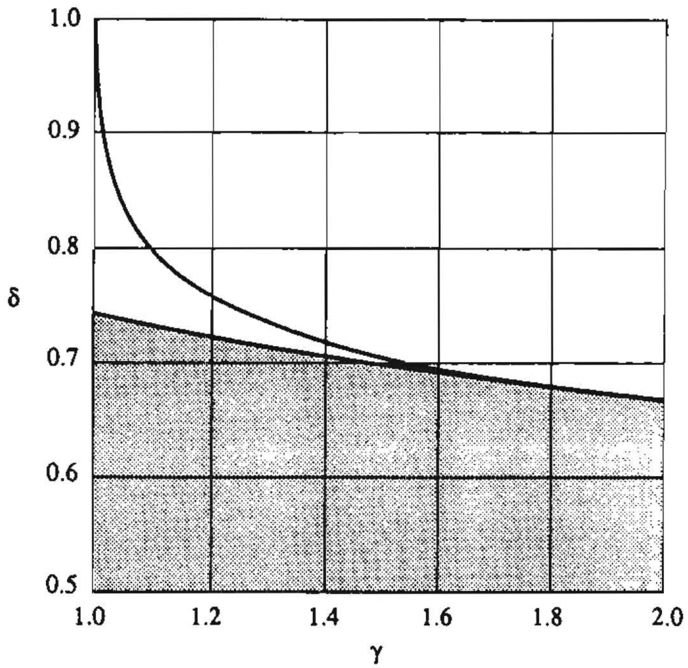

FIGURE 32 The eigenvalue δ as a function of γ for $\nu = 3$. The line that borders the gray region is $\delta = \delta_+$.

the origin of coordinates, that is represented by the saddle point D ($V_D = 1/\gamma$, $Z_D = \infty$, see the preceding Section).

The profiles of the velocity, density, temperature, and pressure are represented in Figures 34–37 respectively, for $t < 0$, $t = 0$ and $t > 0$. In Figure 38 the trajectories of the converging (S) and the outgoing (RS) shocks, as well as some C_- characteristics (that were calculated using (140)) are represented in a x-t diagram. Figure 39 shows some particle trajectories, obtained by integration of (138).

Various properties of the solution can be appreciated in these graphs, that can be summarized as follows (the numerical values are for $\nu = 3$, $\gamma = 7/5$, and are the results of the present calculations):

(a) The imploding shock accelerates continuously and is strengthened as it converges to the center. At the same time, energy concentrates near the shock front as the temperature and pressure there

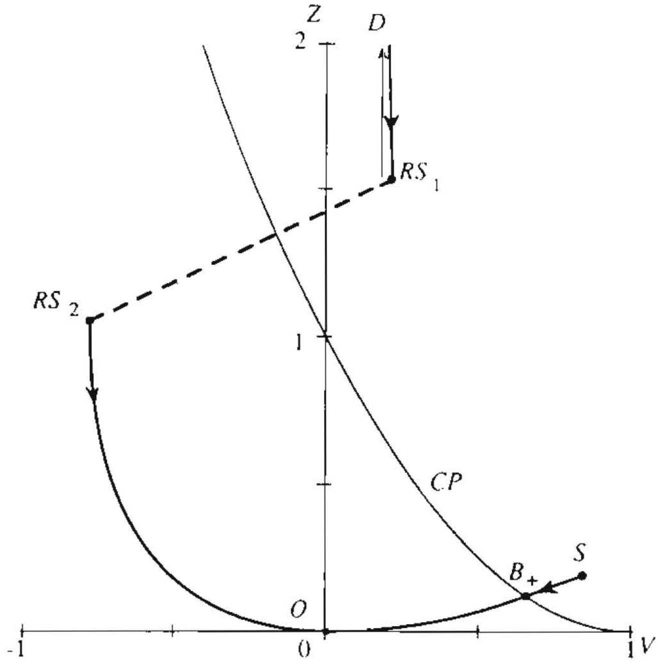

FIGURE 33 Integral trajectory for the convergent shock wave problem ($\nu = 3$, $\gamma = 7/5$).

increase without limit. Notice however that the size of the self similar region decreases with time, with the net effect that the total energy contained in the self similar region actually decreases (Zel'dovich and Raizer, 1967).

(b) Individual gas elements implode with an almost constant velocity. The gas velocity behind the reflected shock is directed outwards, while the gas in front of it is still flowing inwards.

(c) The Mach number at the instant of collapse, $\mathscr{M}_\infty = \lim_{v \to 0} V/\sqrt{Z}$, is uniform and characterizes the solution; it is a diminishing function of γ ($\mathscr{M}_\infty = 1.554$).

(d) The reflected shock has a constant strength.

(e) The pressure in the central region behind the reflected shock is roughly constant near the center and increases slightly towards the shock front.

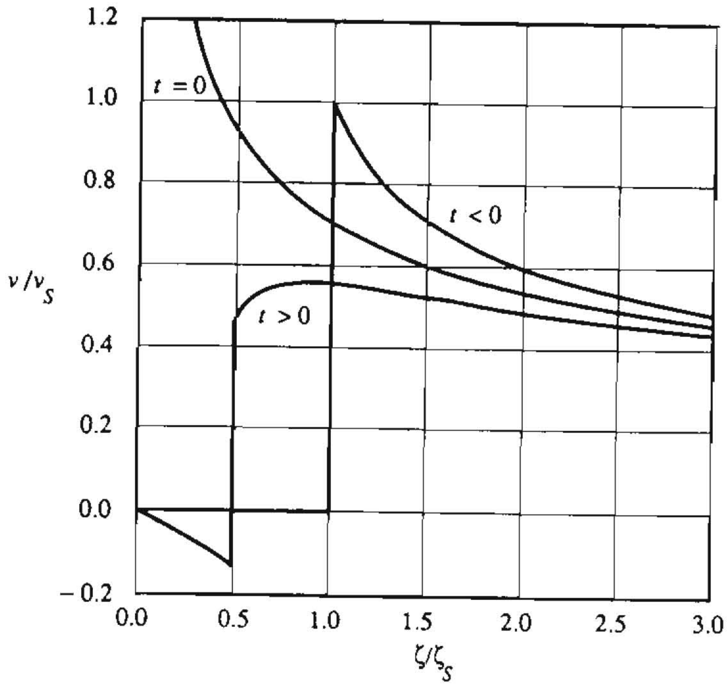

FIGURE 34 Velocity profiles for the convergent shock wave problem ($\nu = 3$, $\gamma = 7/5$).

(f) The temperature diverges in the center of symmetry behind the reflected shock.

(g) For $t < 0$ the gas is compressed by the imploding shock (6 times), and undergoes an additional adiabatic density increase due to the convergence of the flow. As $x \to \infty$ the density tends to a finite value ρ_∞ that stays constant in time; at $t = 0$, $\rho = \rho_\infty$ everywhere ($\rho_\infty/\rho_0 = 20.07$, so that a roughly $3.5\times$ adiabatic compression occurs). For $t > 0$ the imploding gas in front of the reflected shock is further compressed adiabatically from ρ_∞ at $x = \infty$ to a constant value ρ_f just in front of the outgoing shock ($\rho_f/\rho_0 = 64.32$, again a roughly $3\times$ density increase). The density vanishes at the center and rises to a value ρ_m just behind the reflected shock ($\rho_m/\rho_0 = 145.08$). This is the maximum density attained in the process, and remains constant as the

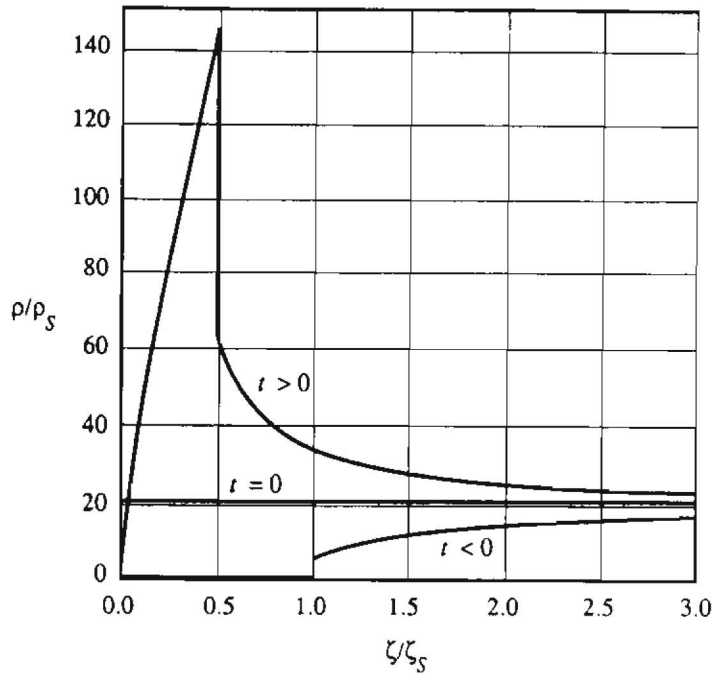

FIGURE 35 Density profiles for the convergent shock wave problem ($\nu = 3$, $\gamma = 7/5$).

shock propagates outwards.† The maximum compression is infinite for $\gamma = 1$ and decreases with increasing $\gamma > 1$ (Lazarus and Richtmeyer, 1977, Rodriguez and Linan, 1978). For comparison, the following values are reported by Meyer-ter-Vehn *et al.*, (1982) for the case $\gamma = 5/3$, $\nu = 3$: $\mathcal{M}_\infty = 0.956$, $\rho_\infty/\rho_0 = 9.47$, $\rho_m/\rho_0 = 32.0$; in this case there is a $4\times$ compression due to the imploding shock.

The limiting characteristic LC (see Section 5.4) represented by the singular point B_+ (or B_-) on the CP through which the integral curve passes is shown in the x, t diagram of Figure 38. It can be appreciated that it divides the converging flow in two regions (I and II):

(a) Region I is represented by the piece SB_+ of the integral curve and corresponds to points between the shock front and the LC. In this

†Zel'dovich and Raizer (1967) report $\rho_\infty/\rho_0 = 21.6$, $\rho_m/\rho_0 = 137.5$ for $\gamma = 7/5$, $\nu = 3$.

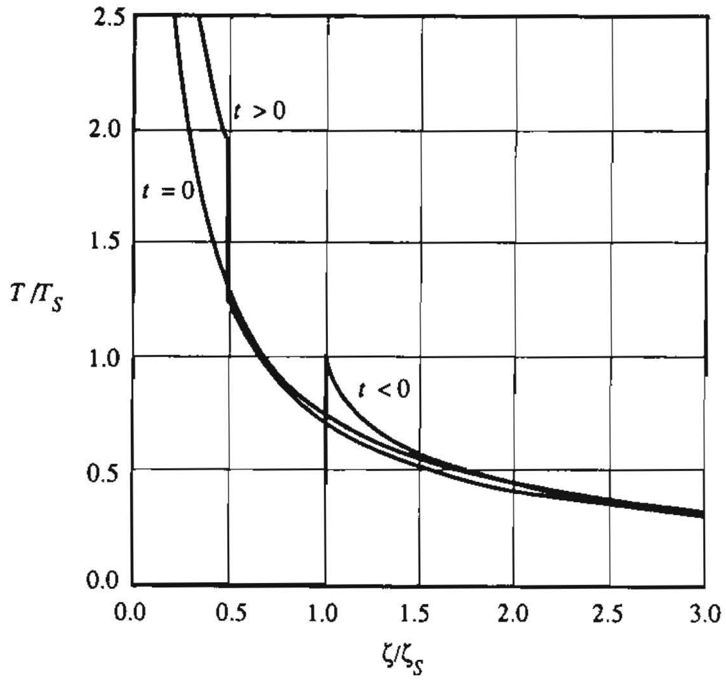

FIGURE 36 Temperature profiles for the convergent shock wave problem ($\nu = 3$, $\gamma = 7/5$).

region the flow is subsonic and all the C_- characteristics, like C' in the Figure will eventually intersect the shock trajectory S. The flow in this region is in causal contact with the gas at $x = 0$, $t = 0$.

(b) Region II is represented by the piece $B_+ O$, and corresponds to points outside the LC in Figure 38. The flow is supersonic and the C_- characteristics, like C'' arrive at $x = 0$ for $t > 0$. Hence the flow in region II is not in causal contact with the gas at $x = 0$, $t = 0$. This means that the collapse of the converging shock front will proceed in the same fashion, regardless of any perturbation that might occur in this region.†

Comparing the particle trajectories of Figure 39 with Figure 38 it can

†Of course such a perturbation will affect the gas at $x = 0$ at some later time $t > 0$.

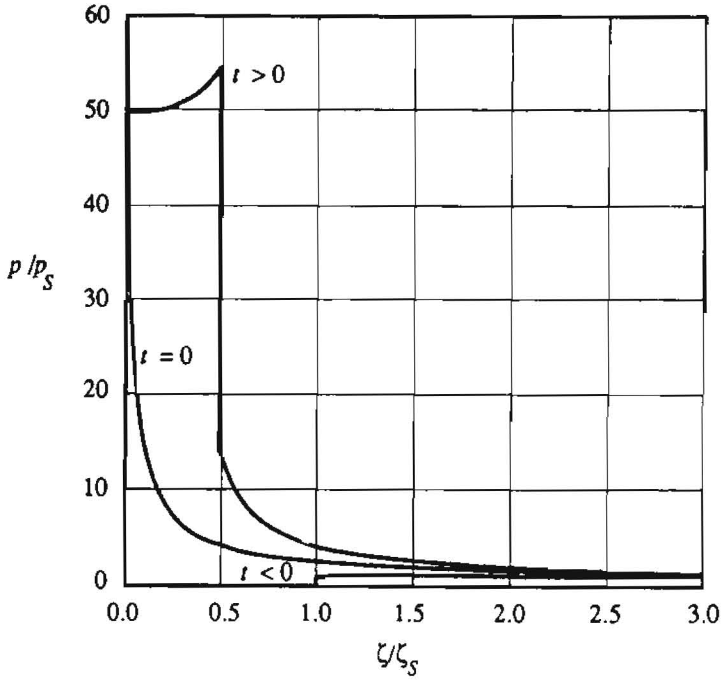

FIGURE 37 Pressure profiles for the convergent shock wave problem ($\nu = 3$, $\gamma = 7/5$).

be appreciated that as t increases and approaches $t = 0$, the portion of gas inside the LC (i.e., region I) shrinks and becomes vanishingly small as more and more gas elements cross the LC passing to region II.

It should be observed that for $t > 0$ the state of the gas in the central region behind the reflected shock is very much like that corresponding to an explosion (the profiles of Figures 34–37 and those of Figure 25 are qualitatively very similar). In fact, the concentration of energy at the origin at the moment of collapse of the convergent shock leads to a state of affairs in a certain sense equivalent to the initial condition of an explosion, i.e., a very high concentration of energy in a small region of space. However the analogy between the two problems cannot be carried too far, considering the different kind of self similarity, that leads to values of δ (and in consequence to motions) very dissimilar.

The numerical solution of the problem of a converging non self

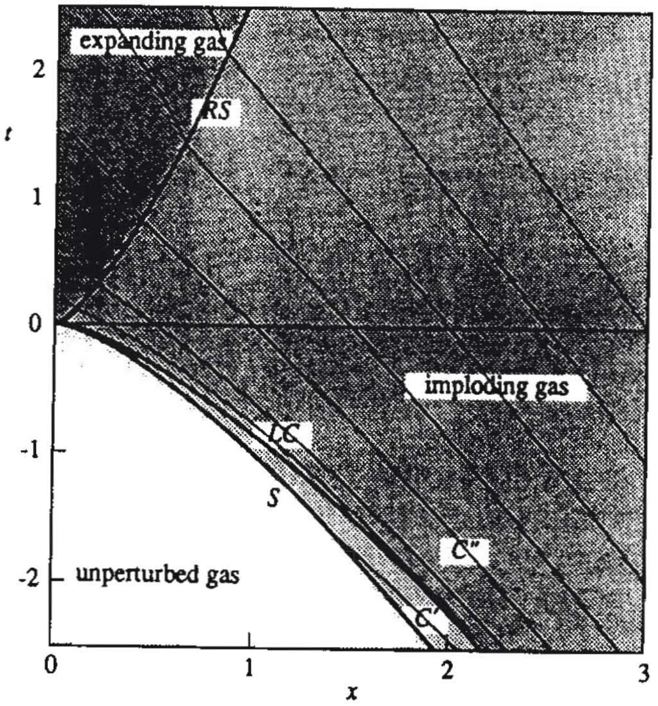

FIGURE 38 Diagram x–t showing the trajectories of the converging shock (S), the reflected shock (RS), the limiting characteristic (LC), as well as some other C_- characteristics ($\nu = 3$, $\gamma = 7/5$); x and t are in arbitrary units.

similar shock wave starting from non self similar initial conditions shows a transition to the self similar asymptotics in good agreement with the theory (Nakamura, 1983). The effect of counterpressure was studied by Welsh (1967). Discussions about the stability of self similar implosions can be found in the papers of Book and Bernstein (1979).

Second kind self similar solutions have been also found in related problems such as the flow into a cavity (Hunter, 1963) and other implosions, see also Meyer-ter-Vehn and Schalk (1982) and the references quoted therein; the problem of the transition to the self similar asymptotics of the flow into a cavity has been investigated numerically by Thomas *et al.* (1986).

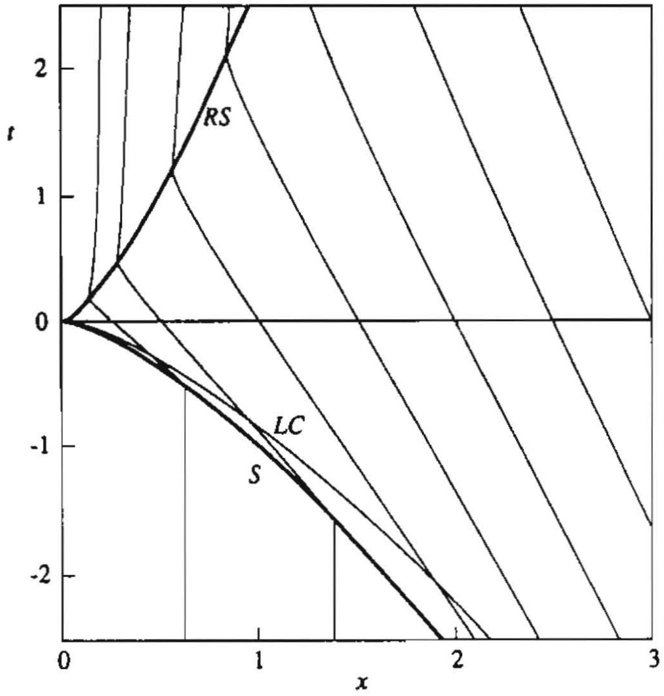

FIGURE 39 Particle trajectories for the convergent shock wave problem ($\nu = 3$, $\gamma = 7/5$); x and t are in arbitrary units.

6 SELF SIMILAR SHALLOW WATER GRAVITY CURRENTS

Shallow water theory is closely analogous to gas dynamics, so that it is convenient to comment briefly the similarity solutions that have been studied in this context.

Consider the equations of shallow water theory:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + x^{-n}\frac{\partial}{\partial x}(x^n hu) = 0, \qquad (171)$$

in which $u(x, t)$ denotes the (horizontal) velocity of an inviscid liquid that flows over a planar horizontal bottom, $h(x, t)$ is the thickness of the current, g is the acceleration of gravity, and the geometrical index n takes the values 0, 1, according if the current is planar, or axially

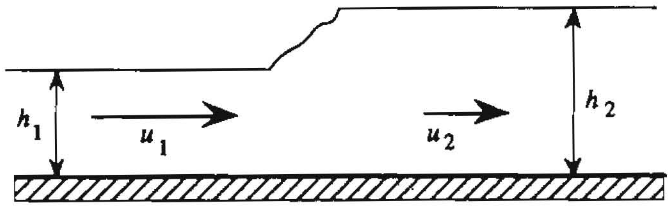

FIGURE 40 An hydraulic jump.

symmetric. We notice that the same equations also describe a gravity current of a dense fluid that intrudes under a less dense ambient fluid, provided one replaces g by $g' = g(\rho - \rho_a)/\rho_a$, where ρ_a is the density of the ambient fluid.

It is well known that eqs. (171) are formally equivalent to the equations of gas dynamics (114) for $\gamma = 2$ (see for example Landau and Lifschitz, 1959b); to show the equivalence it suffices to establish the formal correspondence

$$\rho h \rightarrow \rho, \int_0^h p\, dy \rightarrow p, \tag{172}$$

and one passes from (171) to (114). Clearly, many of the results that one obtains from the dynamics of gases with $\gamma = 2$ can be applied to the shallow water gravity currents, with the qualification that the matching conditions at discontinuities are different. Discontinuous solutions must now be joined according to the matching conditions for hydraulic jumps, as we shall indicate below.

The self similar solutions shall now be of the form

$$u = \frac{\delta x}{t} V(\zeta),\ gh = \left(\frac{\delta x}{t}\right)^2 Z(\zeta),\ \zeta = \frac{x}{bt^\delta}, \tag{173}$$

in which V, Z satisfy the autonomous differential equation (125) with $\gamma = 2$, and $\zeta(V)$ is obtained by integration of (126).

In this type of flows, discontinuities of h and u, that are called hydraulic jumps, may appear. The hydraulic jumps play here a role analgous to that of shock waves in gas dynamics. The matching conditions for hydraulic jumps can be found in the textbooks on hydrodynamics (see for example Landau and Lifschitz, 1959b), and are derived by requiring conservation of mass and momentum across the discontinuity (see Figure 40). If the suffixes 1, 2 denote the variables at both

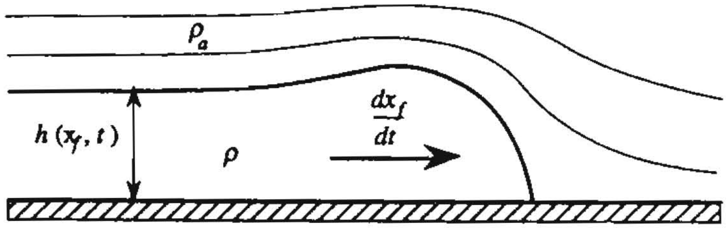

FIGURE 41 The head of a gravity current that is intruding in an ambient fluid.

sides of the jump, one obtains

$$Z_2 = \frac{\sqrt{1 + 8\mathscr{F}} - 1}{2} Z_1,\ V_2 - 1 = \frac{2}{\sqrt{1 + 8\mathscr{F}} - 1}(V_1 - 1),$$

$$\mathscr{F} = \frac{(V_1 - 1)^2}{Z_1}, \tag{174}$$

in which $\mathscr{F} = u_1^2/gh_1$ is the Froude number (that plays in the shallow water theory a role analogous to that of the Mach number in gas dynamics). The critical parabola is given as in the case of gas dynamics by $Z = (V - 1)^2$, and corresponds to $\mathscr{F} = 1$. Points of the phase plane above the CP correspond to subcritical flow ($\mathscr{F} < 1$), and those below it represent supercritical flow ($\mathscr{F} > 1$).

In the case of a current that intrudes in an ambient fluid, we shall have internal hydraulic jumps. The matching conditions will be in general more complicated (see Yih, 1965), but when the depth of the ambient liquid is infinite the corresponding formulae are formally identical to those of an ordinary hydraulic jump, with the substitution $g' \rightarrow g$ in the definition of the Froude number.

If a gravity current intrudes in an ambient fluid, its head will encounter a resistance to its advance, and will assume a shape such as is sketched in Figure 41 (see for example Von Kármán, 1940, Brooke Benjamin, 1968, Simpson and Britter, 1979, Grundy and Rottman, 1986, etc.). If we denote by x_f the position of the front, the boundary condition that takes into account the resistance of the ambient fluid is of the form

$$\beta^2 g' h(x_f, t) = \left[\frac{dx_f(t)}{dt}\right]^2, \tag{175}$$

In which β is a constant parameter of the order of unity for liquids whose densities are not very different. Then the current has a finite thickness at its front. The case in which the ambient fluid has a vanishing density corresponds to the limit $\beta \to \infty$ in (175), then $h \to 0$ at the current front. In terms of the phase variables the boundary condition (175) takes the form

$$V(\zeta_f) = 1, \beta^2 Z(\zeta_f) = 1. \tag{176}$$

There are not many papers in the literature about self similar gravity currents. We can mention the classical problem of the breaking of a dam (see for example Whitham, 1974) that is entirely analogous to the expansion of a gas into a vacuum that we discussed in Section 5.4, and the current produced by the discharge of a constant volume of fluid whose the scaling laws have been obtained by Fannelop and Waldman (1972) and Hoult (1972). Rottman and Simpson (1983) have investigated experimentally this type of current. A numerical study of the approach to self similarity, and of the stability of these solutions has been made by Grundy and Rottmann (1985). Huppert and Simpson (1980), and Maxworthy (1983), have performed experiments on currents whose volume varies with time according to a power law of the type t^α.

A theoretical investigation of the self similar gravity currents has been made by Britter (1979) in the case of constant volume ($\alpha = 0$) and more general cases were studied by Grundy and Rottman (1986) with the phase plane formalism; in this reference currents whose volume varies with time ($\alpha \neq 0$), that intrude into an ambient fluid (with the boundary condition (176)), were considered. This paper is the most extensive in what respects the theory, but several results are not satisfactory: in the case $n = 0$, the authors cannot obtain self similar solutions for certain intervals of values of the parameter β, and for axial symmetry they do not find any self similar solution at all. These negative results (which are not adequately explained in the paper) are contrary to intuition, and at best of doubtful validity. In fact, this Author has investigated these matters and was able to find self similar solutions in the cases in which Grundy and Rottman (1986) did not succeed (these results will be published elsewhere).

The scaling laws for currents whose volume varies with a power of time are obtained from (173) with

$$\delta = \frac{2 + \alpha}{3 + n}. \tag{177}$$

More details can be found in the above mentioned references.

7 SELF SIMILAR CREEPING FLOWS

Viscosity dominated gravity currents occur in many instances of interest for geophysics, geology, engineering, and environmental sciences. The main characteristic of these flows is that the motion is essentially horizontal, and is governed by a balance between gravity and viscous forces, inertia effect being negligible (see Huppert, 1986, Kerr and Lister, 1987, Simpson, 1982, Hoult, 1972, etc.). In general, except perhaps at the beginning of the phenomenon, the length of the current is much greater than its thickness; with these assumptions, the flow can be described by means of the so called lubrication theory approximation (Buckmaster, 1977, Huppert, 1982).

Some self similar solutions representing this type of currents have been studied by Huppert (1982), who considered flows with planar and axial symmetry over a rigid horizontal supporting surface; it was assumed that the volume of the current varied according to a power law of the time. Britter (1979), Didden and Maxworthy (1982), Huppert (1982), and Maxworthy (1983) performed experiments in this field. The complete theory of the self similar solutions for these phenomena, based on a generalization of the Sedov–Courant–Friedrichs phase plane formalism was developed by Gratton and Minotti (1990), who also discussed a considerable number of new solutions.

The interest in this type of phenomena goes far beyond the topic of viscous gravity currents, because, as we shall show below, the nonlinear parabolic equations (in contrast to the equations of gas dynamics and of the shallow water theory, which are hyperbolic) that govern these flows are mathematically equivalent to those that describe many other important physical phenomena (Seshadri and Na, 1985); the list is long, and it includes nonlinear diffusion, nonlinear heat conduction (transport of heat by radiation in ionized gases, electron conduction in plasmas, etc.), the Dupuit–Forchheimer equations for groundwater flow in unconfined aquifers, and the porous media equation (see for example Peletier, 1981); other phenomena are mentioned in Bear (1982), Boyer (1962), and Lacey *et al.* (1982). In this context it can be mentioned that the solutions found by Pattle (1959) for nonlinear diffusion, and the self similar solutions of the nonlinear heat diffusion equations discussed by Barenblatt (1979), Tappert (1977), and Barenblatt and Zel'dovich (1972), are analogous to some of the solutions that describe self similar viscous gravity currents derived by Huppert (1982) and by Gratton and Minotti (1990). The stability of some of the self similar solutions of this family has been investigated by Gratton and Minotti (1988).

The theory has also been extended to non newtonian liquids that follow a power law rheology, that are of interest for applications to flows in the Earth's lithosphere (Gratton *et al.*, 1988); a further extension is due to Diez (1989) and deals with the self similar solutions of the problem of water percolation in soils.

We now summarize the theory (Gratton and Minotti, 1990). Let $H(x, t)$ be the thickness of the current, and $v(x, z, t)$ the velocity, which we shall assume is nearly horizontal; here x denotes the horizontal coordinate (cartesian or radial, according if planar or axial symmetry is assumed), z is the vertical coordinate; the boundary condition at the bottom will be $v(x, z = 0, t) = 0$, and at the free surface we shall require $(\partial v/\partial z)_{z=H} = 0$. Using these conditions, and with the above assumptions one can derive the vertical velocity profile:

$$v = \frac{3uz}{H^2}(2H - z), \tag{178}$$

in which $u(x, t) = 2v(x, z = H, t)/3$ is the average horizontal velocity. It is important to notice that the only constant dimensional parameter that enters in the governing equations of the problem is g/ν (ν denotes the kinematic viscosity coefficient), and that it can be absorbed defining a new dependent variable that replaces H, as

$$h = (g/3\nu)^{1/3} H. \tag{179}$$

One then obtains the basic equations of the problem in the form

$$h^2 \frac{\partial h}{\partial x} + u = 0, \quad \frac{\partial h}{\partial t} + x^{-n} \frac{\partial}{\partial x}(x^n uh) = 0. \tag{180}$$

in which no constant dimensional parameter appears. These equations are the starting point for the introduction of the phase plane formalism. In (180) the geometrical index n is 0 for planar, and 1 for axial symmetry.

It is possible to use the first of the eq. (180) to eliminate u thus

obtaining a single second order equation of the general form

$$\frac{\partial h}{\partial t} = x^{-n} \frac{\partial}{\partial x}\left(x^n h^m \frac{\partial h}{\partial x}\right), \tag{181}$$

in which the nonlinearity index takes the value $m = 3$ in the present case. When the governing equation is written in the form (181), it can be recognized the analogy between the problem of viscous gravity currents and the nonlinear diffusion, nonlinear thermal conduction, and other phenomena of the same mathematical nature that we mentioned above. Heat conduction by radiation in a fully ionized gas corresponds to $m = 13/2$; electron heat conduction in a plasma to $m = 5/2$; with $m = 1$ one obtains the Dupuit–Forchheimer equation, etc. Since some of these problems are by nature three dimensional, the case $n = 2$, that corresponds to spherical symmetry is also physically meaningful (this is not true for the viscous currents nor for the groundwater flow, that are essentially two-dimensional phenomena). We notice also that the linear case ($m = 0$) cannot be studied by means of the phase plane formalism we shall presently develop, as it is not possible to eliminate from the governing equations the dimensional parameter (the diffusivity) by means of the substitution (179); if $m = 0$, this parameter necessarily enters in the definition of the self similar variable (see Sections 3.1 and 4.2).

The eqs. (180) involve only quantities having the dimensions of length, time, or combinations thereof, then h and u can be expressed as

$$h = (x^2 t^{-1} Z)^{1/3}, \quad v = x t^{-1} V, \tag{182}$$

in which the phase variables Z, V are in general dimensionless functions of x, t and the constant parameters involved in the initial and/or boundary conditions. The problem will be self similar if these conditions introduce no more than a single parameter with independent dimensions. We shall assume this to be the case; we call b this parameter, and assume that its dimensions are given by

$$[b] = [x t^{-\delta}], \tag{183}$$

in which δ is a numerical constant. The motion is then self similar, and Z, V will be functions of the self similarity variable

$$\zeta = \frac{x}{b t^{\delta}}. \tag{184}$$

Substitution of (182) and (184) in (180) yields a system of two ordinary differential equations for the phase variables, that we shall not write down for brevity. From this system one can obtain the basic equations of the phase plane formalism in the form of an autonomous equation for $V(Z)$:

$$\frac{dV}{dZ} = \frac{(2\delta - 1)Z + 3(1 + n)VZ + 3(\delta - V)V}{3Z(2Z + 3V)}, \quad (185)$$

whose solutions are the integral curves, and a second equation:

$$\frac{d}{dZ}(\ln|\zeta|) = -\frac{1}{2Z + 3V}. \quad (186)$$

that allows to calculate by means of a quadrature the self similar variable in terms of the phase variables, once the adequate integral curve $V(Z)$ has been found.

For the general nonlinear diffusion equation (181) one obtains following a similar procedure:

$$\frac{dV}{dZ} = \frac{(2\delta - 1)Z + m(1 + n)VZ + m(\delta - V)V}{mZ(2Z + mV)}, \quad (187)$$

and:

$$\frac{d}{dZ}(\ln|\zeta|) = -\frac{1}{2Z + mV}, \quad (188)$$

in place of (185) and (186) (Gratton and Minotti, 1990); in this case $n = 0, 1, 2$ for planar, cylindrical, and spherical symmetry, respectively.

It can be observed that the autonomous differential equation (185) is simpler than its counterpart (125) for gas dynamics since the numerator and the denominator of its r.h.s. are polynomials of the second degree in V, Z, while those of (125) are of the fourth degree. There are in the present case six singular points, namely: $O[Z_O = 0,\ V_O = 0]$, $A[Z_A = 0,\ V_A = \delta]$, $B[Z_B = -3/2(5 + 3n),\ V_B = 1/(5 + 3n)]$, $C[Z_C = 0,\ V_C = \infty]$, $D[Z_D = \infty,\ V_D = (1 - 2\delta)/3(1 + n)]$, and $E[Z_E = \infty,\ V_E = \infty]$. Their properties and those of the family of the self similar viscous gravity currents has been analyzed in depth in the paper of Gratton and Minotti (1990) to which the reader is referred. There is a certain parallelism between this family and that of the dynamics of gases, notwithstanding several important differences. One of these is that here the whole phase plane (and not only the $Z > 0$ half plane, as before) is physically meaningful: the solutions with $Z > 0$ correspond to $t > 0$, and those with $Z < 0$ are meaningful for $t < 0$. An other important difference is that in the present case there is nothing equivalent to the critical parabola, nor there are solutions with jumps, or shocks, as it is to be expected given the diffusive nature of the governing equations.

An interesting property of the currents we are considering is the occurrence of sharp, well defined fronts (that correspond to the finite propagation speed of a thermal wave in the case of nonlinear heat conduction), but perhaps the single most striking and novel feature is the occurrence of solutions with a front that does not move during a finite amount of time, remaining at a fixed position, while nevertheless motion is taking place behind it; these solutions are usually called waiting-time solutions in the literature on nonlinear diffusion.

In what follows we shall discuss some of the solutions that can be obtained from eq. (180), to illustrate some of the most remarkable features of the self similar solutions of this type of nonlinear parabolic equations.

7.1 The Breaking of a Dam Containing A Viscous Liquid

Let us assume that a vertical wall at $x = 0$ (we consider planar symmetry) supports a semi infinite pool of a viscous liquid in the $x < 0$ region, whose depth is H_0. In a certain moment, that we shall take as $t = 0$, the wall is removed and the liquid overruns the $x > 0$ region. There is a single constant dimensional parameter, $h_0 = (g/3\nu)^{1/3} H_0$, so that the problem is self similar in the variable

$$\zeta = \frac{x}{h_0^{3/2} t^{1/2}}; \quad (189)$$

i.e.:

$$b = h_0^{3/2},\ \delta = \tfrac{1}{2}. \quad (190)$$

The relevant part of the phase plane is represented in Figure 42. Since initially no fluid is present in the $x > 0$ half plane, the solution must present a front at a finite distance from the origin, that moves in time; The integral curve that represents the flow for $x > 0$ is labeled A in the Figure; it starts at the singular point E† that represents the origin of

†The singular point E is a node.

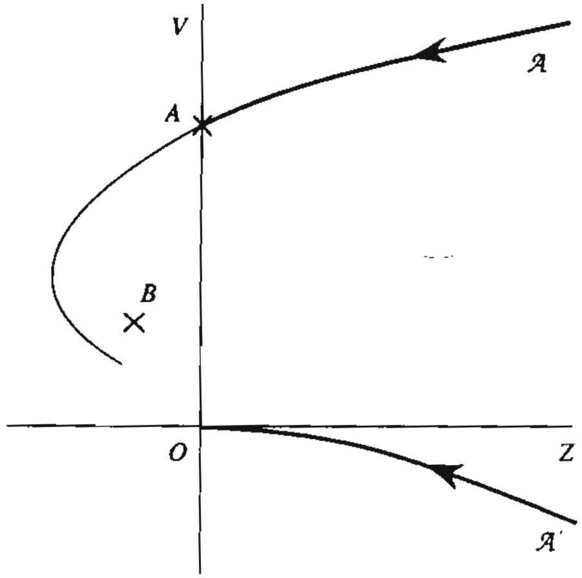

FIGURE 42 The integral curves for the dam breaking problem.

coordinates ($x = 0$), and arrives at the singular point A, that represents the front (notice that as A is a saddle, the integral curve A is unique). For $x < 0$ the flow is represented by the curve A', that joins the origin O of the phase plane (that represents the point $|x| = \infty$ of the fluid) with E. It can be shown that O is a saddle-node, there being an infinite number of integral curves arriving at O from the $Z > 0$ half plane, but only a single curve that arrives there from the $Z < 0$ half plane; then A' must be determined imposing some additional conditions, which obviously can be no other than the requirement of the continuity of h and u at $x = 0$ (i.e., at the point E); these conditions allow to determine the curve A' and the integration constant that fixes ξ_f (the front position).

Near A, i.e. near the front, the following asymptotic formulae hold for the integral curve A:

$$V = \delta + \frac{(5 + 3n)\delta - 1}{12\delta} Z, \quad Z = 3(1 - \zeta/\zeta_f), \tag{191}$$

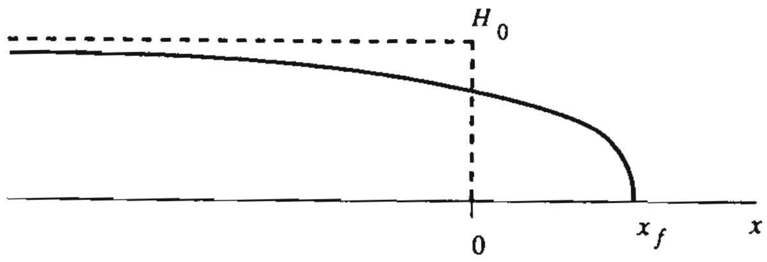

FIGURE 43 Qualitative profile of the viscous current produced by the breaking of a dam.

$$H \sim t^{(2\delta - 1)/3}(1 - x/x_f)^{1/3}, \quad u \sim \frac{x}{t},$$

Near E, which as said represents the point $x = 0$, we have the following formulae for the curves A, A'

$$Z \sim V^2, \quad Z \sim \zeta^{-2}, \quad H \sim t^{(2\delta - 1)/3}, \quad u \sim t^{\delta - 1}. \tag{192}$$

Finally, all the integral curves that arrive at O have the following asymptotic behavior ($\delta \neq 0$), that describes what happens near $x \to -\infty$:

$$V = -\frac{2\delta - 1}{3\delta} Z\left[1 - \frac{(5 + 3n)\delta - 4}{3\delta^2} Z + \cdots\right], \quad Z \sim |\zeta|^{-1/\delta},$$

$$H \sim \text{const.}, \quad u \sim \frac{x}{t} e^{-K\zeta^2}, \tag{193}$$

in which K denotes a constant.

The profile of the solution has been sketched in Figure 43. It can be observed that the current is characterized by a constant vertical scale H_0, so that the only change of its profile with the passing of time is a horizontal stretching proportional to $t^{1/2}$.

7.2 Waiting-time Solutions

We shall consider the singular point B, located at

$$Z_B = -\frac{3}{2(5 + 3n)}, \quad V_B = \frac{1}{5 + 3n}. \tag{194}$$

It can be shown that B can be either a node, or a focus, according to the values of δ and n. In addition, for any δ, n,

$$V = V_B, Z = Z_B, \tag{195}$$

give an exceptional exact solution of the governing equations (180) that is represented in the phase plane by the single point B. This solution corresponds to negative times, and is given by

$$H = -\left[\frac{9}{10+6n}\frac{\nu x^2}{gt}\right]^{1/3}, u = \frac{1}{5+3n}\frac{x}{t}, t < 0. \tag{196}$$

It describes a current with a fixed front at $x = 0$ whose profile varies as $x^{2/3}$ and whose thickness at a fixed position grows infinitely as $-t \to 0$. The velocity of the current is zero at the front for any given time, and increases linearly with x: for fixed $x(\neq 0)$ it increases as the inverse power of time, and is always directed towards the front. The equation of motion of a parcel of the liquid that is moving with the average velocity is

$$x = \text{const.}(-t)^{1/(5+3n)}. \tag{197}$$

Physically, this means that given adequate conditions, and without any obstacle to prevent its advance, there can be currents with a front that waits for a finite amount of time while the flow behind reorders itself, before starting to move. This type of behavior is typical of the so-called waiting type solutions, that appear in problems governed by nonlinear diffusion-type equations such as (181). There have been recently quite a few papers on this intriguing topic (among others see for instance Lacey *et al.*, 1982, Kath and Cohen, 1982, Smyth and Hill, 1988), that is by itself an interesting field of research. The problem of finding the appropriate continuation of these solutions for $t > 0$, when the front begins to move (and the present treatment breaks down) has been considered by Lacey *et al.* (1982).

The exceptional solution we have discussed is not the only one that exhibits a waiting-time behavior; there are several integral curves in the phase plane that describe solutons of this type; they are curves related to the singular point B, and, for $\delta = 0$, to the point O (Gratton and Minotti, 1990).

The peculiar behavior of a waiting time solution can be observed most easily be means of a very simple experiment that can be performed with a flat bottomed vessel that contains a thin layer of some very viscous fluid (such as some silicones). To set up an initial condition that leads to the desired flow, lift one of the sides of the vessel and wait until the liquid settles under the action of gravity, taking a wedge shaped profile; then lower rapidly the side so that the bottom of the vessel is horizontal. In this way an initial condition is established, in which the liquid rests on a flat horizontal surface, and has a depth that varies linearly with the position, being zero at the front. It will then be observed that the subsequent motion consists of a change of the shape of the profile, but the front itself does not move. This will go on for a certain interval of time, during which the profile near the front acquires a bulging shape, until suddenly the front begins to advance. The type of motion we have described corresponds to a waiting time solution. The interpretation is as follows: to advance, the profile of the liquid must have the unique shape given by (191) in the vicinity of the front. If the initial condition is such that the profile of the current does not agree with the 1/3 power law (191) in the vicinity of the front, there cannot be a motion of the front itself. The flow that ensues in this case takes place behind the front, and gradually distorts the profile so that eventually it acquires that particular shape that is compatible with the movement of the front. Then, there must be a first stage of the phenomenon in which the front sits still for a finite time interval, waiting for the flow behind to reorganize. The self similar waiting time solutions describe the very last moments of this first stage.

7.3 The Collapse of a Converging Viscous Current

Let us consider an axially symmetric viscous current that converges towards the center of symmetry. Such a current can be produced if there is initially a pool of liquid outside a circular vertical retaining wall (the region inside the wall being dry), and at a certain moment the wall is removed letting the liquid run towards the center. The current will have a converging front whose radius will decrease with time, and that will finally collapse at the center. We shall focus our attention on the last stages of the phenomenon, near the moment of collapse, and shall be interested in the properties of the flow near the front, for small values of the radius as compared to any constant parameter characteristic of the initial conditions (for example, the radius of the retaining wall). In this situation no characteristic constant governing parameter remains, because those that appear in the initial conditions are no longer adequate as scales of the properties within the region of interest, and the characteristic parameters of the flow within this region are functions of time. Consequently the flow must be self similar, but the self similarity exponent δ cannot be found by means of dimensional analysis. Therefore we are in the presence of a case of self similarity of the second kind. We shall now proceed to find the solution by direct construction.

Since the current has a moving front, the solution must be represented by the single integral curve passing through the singular point A.† We recall that we are concerned with what happens before the collapse of the front (which we shall take to occur at $t = 0$), i.e., for $t < 0$, which means that the integral curve we are seeking must lie in the $Z < 0$ half plane. The integral trajectory leaving A must end at some other singular point; there are in the present case only three possibilities: the points O, B, and C. It is easy to see that B and C must be discarded, as an integral curve from A to any of these points represents a current whose thickness tends to zero, or to infinity, as $-t \to 0$, at any finite distance behind the front (to see this it is necessary to investigate the behavior of the solutions in the neighborhood of the singular points concerned; we skip the details for brevity, the interested reader can find them in the paper by Gratton and Minotti, 1988a).

We conclude that the current must be represented by an integral curve joining A with O (more precisely, a portion of this curve), that has indeed the desired properties, viz., it describes a flow that for $-t \to 0$ has H, u, finite and non vanishing at any finite distance behind the front. But O is (like A) a saddle for curves approaching it in the $Z < 0$ half plane. Clearly, the single curve leaving A will not in general (i.e., for an arbitrary value of δ) coincide with the single curve arriving at O (see Figure 44). Then, for arbitrary δ there will be no solution. Only for a special value of δ, that we shall denote by δ_c, there will be an integral curve joining A with O, that represents the solution we are looking for. Thus the self similarity exponent is found by solving an eigenvalue problem, as corresponds to a self similarity of the second kind.

By numerical calculation one finds $\delta_c = 0.762\ldots$. Near the front whose position is denoted by x_f the solution is given by

$$H = \left(-\frac{9\nu}{g}\delta_c\frac{x_f^2}{t}\right)^{1/3}\left(1-\frac{x}{x_f}\right)^{1/3}\left[1-\frac{4\delta_c-1}{24\delta_c}\left(1-\frac{x}{x_f}\right)+\cdots\right],$$

†Actually there is a second integral curve passing through A (that is a saddle): it is the line $Z = 0$, that corresponds to a trivial, uninteresting solution.

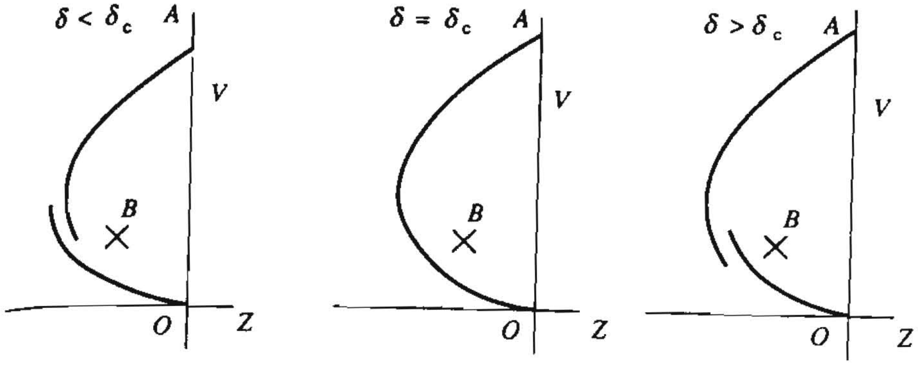

FIGURE 44 The eigenvalue problem for the collapse of a converging viscous current.

$$u = \delta_c\frac{x}{t}\left[1-\frac{8\delta_c-1}{4\delta_c}\left(1-\frac{x}{x_f}+\cdots\right]\right.,\quad x_f = K(-t)^{0.762\ldots},$$
$$K = \text{const.} \tag{198}$$

It can be observed that as $-t \to 0$ the front accelerates, and its velocity becomes infinite the moment of collapse. In Figure 45 the profiles of this current can be appreciated.

For $t > 0$, i.e., after the collapse, the solution must be represented by an integral curve in the $Z > 0$ half plane, corresponding of course to $\delta = \delta_c$; this curve must represent a flow with finite thickness and vanishing velocity at $x = 0$. It can be shown (Diez *et al.*, 1989) that this curve is precisely the single integral curve that leaves the singular point D (which is a saddle) and arrives at O. The profiles of the current after the collapse are shown in Figure 46.

It can be observed that this problem has several features in common with that of the collapse of a cylindrical (or spherical) shock wave (Section 5.7), that leads to the classical solution of Guderley (1942). The generalization of this problem for any n, m, has been investigated by Diez *et al.* (1989). In Figure 47 we present the dependence of the eigenvalue δ_c on the geometrical and nonlinearity indices.

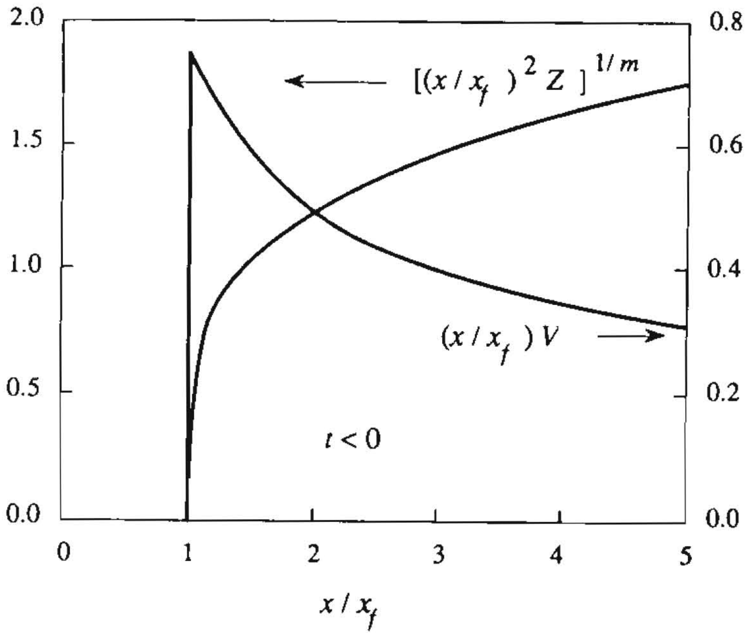

FIGURE 45 Thickness and velocity profiles before collapse of a converging viscous current.

7.4 Progressive Waves and Limiting to Self Similar Solutions

To conclude this Section we shall discuss a special analytic solution of eqs. (180) and (185), that will show the connection between the self similar solutions and the solutions of the progressive wave type.

In the case $n = 0$ the governing equations (180) are translationally invariant, so that they admit solutions of the progressive wave type:

$$h = h(\xi), u = u(\xi), \xi = ct - x, \tag{199}$$

with c = const.. Of course, in this case c is not a characteristic of the medium, but will be determined by the boundary conditions, for example the (constant) velocity of a piston that is pushing the fluid, so that it can take any value.

It is known (see for instance Barenblatt and Zel'dovich, 1972, and Barenblatt, 1979) that there is a close connection between the self similarities and the traveling waves. Actually the traveling waves are

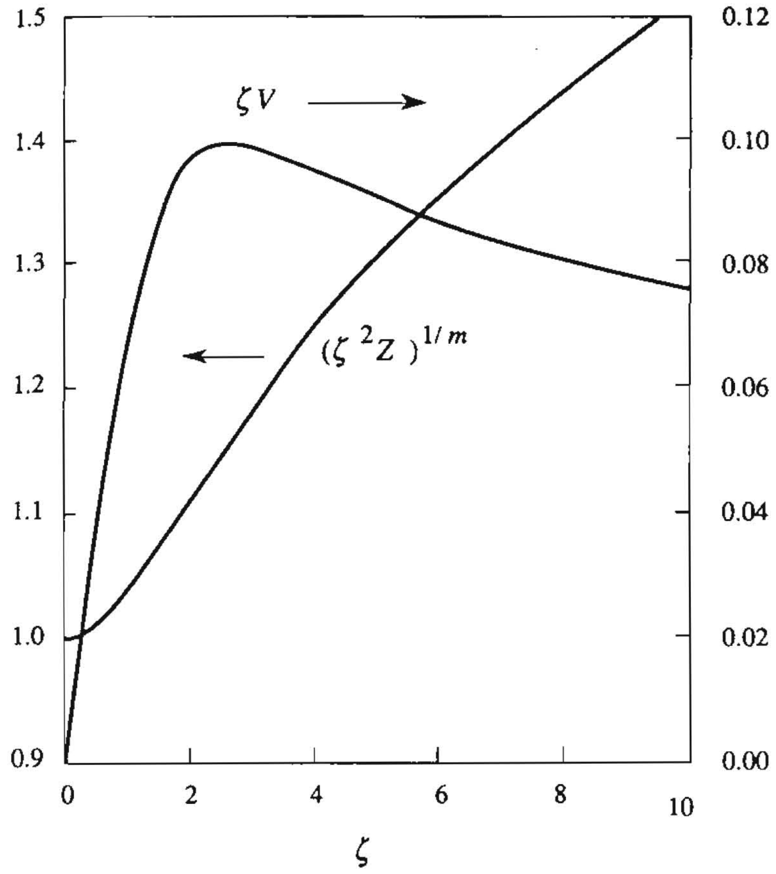

FIGURE 46 Thickness and velocity profiles after collapse of a converging viscous current.

themselves self similar, as can be formally shown performing the substitution

$$\lambda = \frac{\mu}{\tau^{c}}, \mu = e^{-x/L}, \tau = e^{-t/T}, \tag{200}$$

in (199). In (200), λ is the self similarity variable, L and T are two characteristic parameters with dimensions of length and time, respectively, with $L = cT$, and $\xi = \ln \lambda$. This is not a self similarity of the

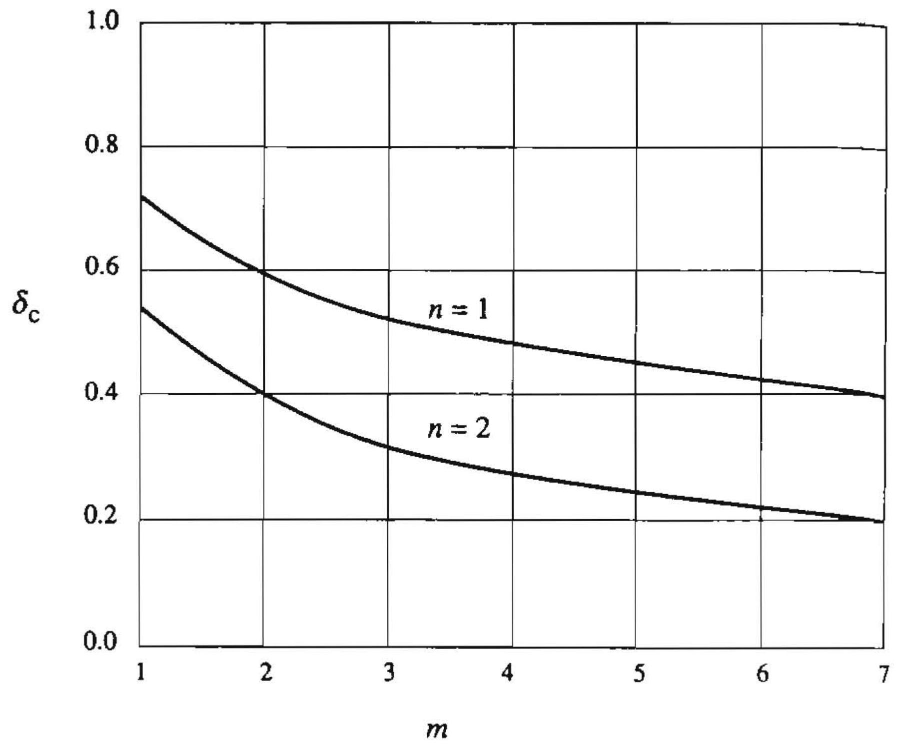

FIGURE 47 Self similar exponents (eigenvalues) for converging currents as function of the nonlinearity index m, for cylindrical ($n = 1$) and spherical ($n = 2$) symmetry.

type we have been studying in the previous Sections (in which the self similar variable is expressed as a monomial of powers of the independent variables), but anyway we can call it by this name, since the solutions (199) do not depend on x, t separately, but only through a particular combination of both. The solutions of the type (199), (200) can be derived from the phase plane formalism by means of a certain limiting process, whose details can be found in the book of Sedov (1959), and because of this they are a special case of a certain class of solutions that are called "limiting to self similar" in the literature. We shall not discuss the details of the general method for the construction of the limiting to self similar solutions, as the interested reader can find it in the above mentioned reference, but instead we shall derive the

of traveling wave solutions for the planar viscous gravity flows, and show their connection to the self similarities of the type (182), (184).

We substitute in eq. (180) the definitions (199), and denote with $'$ the derivative with respect to ξ; then, after eliminating the dependent variable u and performing a first integration we obtain:

$$h^3 h' - ch = K = \text{const.} \,. \tag{201}$$

If $K = 0$ we find the solution

$$H = \left[\frac{9\nu c}{g}(\xi - \xi_0)\right]^{1/3}, \; u = c, \; \xi_0 = \text{const.}, \tag{202}$$

that represents a current that moves with a constant velocity c without changing its profile, and whose front is located at $x_f = ct - \xi_0$. This type of flow is produced by a spatula advancing with uniform speed, pushing a constant volume of liquid in front of it.

For $K \neq 0$ it is also possible to obtain analytical solutions of the eq. (201); we shall not spend time with them, as they are discussed in the paper of Gratton and Minotti (1990), it will suffice to say that they describe traveling waves corresponding to other boundary conditions, such that the volume of fluid pushed by the spatula is not conserved.

It is an interesting fact that the solution (202) is also represented among the self similar solutions of the phase plane. Indeed, the eq. (185) admits, for $n = 0$, $\delta = 1$, an exact analytic solution (that passes through the singular point A) given by

$$Z = 3V(V - 1). \tag{203}$$

It can be verified that (203) represents precisely the solution (202). In contrast, the remaining traveling wave solutions corresponding to $K \neq 0$ are not represented in the phase plane: in other words, they do not admit a self similar representation of the type (182), (184). This is precisely what should be expected, since the traveling waves depend in general on two dimensional parameters, c and K, with independent dimensions; only if $K = 0$ we are left with a single parameter, so that a self similarity of the type (182), (184) is obtained.

8 FINAL REMARKS

The overview of self similarity we have presented in this article is far from being complete and thorough. For the sake of brevity we have left untouched important topics as for instance the problem of the spectrum of exponents of the self similarity variables, certain issues connected with the stability of the solutions, as well as a large number of applications to the mechanics of continuous media, the theory of turbulence, astrophysics, etc.. Various matters that would have perhaps deserved a more detailed and in depth analysis have been very summarily discussed. To assist those who are interested, or need to further study some of the present topics, we have endeavored to give an abundant, albeit with no pretense of completeness, list of references that we believe should be sufficient to orient the reader in this field.

Notwithstanding these limitations I hope that the various examples discussed will have helped the reader to grasp the fundamental concepts involved in self similarity and given him a taste of its richness, as well as a glimpse of the beauty and elegance of the solutions and of the usefulness of the theory.

Acknowledgements

Thanks are due to my collaborators Javier Diez and Fernando Minotti for critical remarks on the manuscipt. I also wish to thank Professor C. Ferro Fontán for many helpful discussions. Part of this paper was written when visiting as an Associate the International Centre of Theoretical Physics at Trieste, Italy, and it is a pleasure to thank Professor Abdus Salam for the support and hospitality while at the ICTP.

This work was supported by grants from the Organization of American States, the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and the University of Buenos Aires.

References

Anasimov, S. I. (1970). *JETP Lett.*, **12,** 287.
Barenblatt, G. I., and Krylov, A. P. (1955). *Izv. Akad. Nauk SSSR, OTN* **2,** 14.
Barenblatt, G. I., and Sivashinksii, G. I. (1969). *Appl. Math. Mech. PMM* **33(5),** 836.
Barenblatt, G. I., and Zel'dovich, Ya. B. (1972). *Ann. Rev. Fluid Mech.*, **4,** 285.
Barenblatt, G. I. (1979). *Similarity, Self-Similarity, and Intermediate Asymptotics*, Consultants Bureau, New York and London.
Barenblatt, G. I., Entov, V. M., and Ryzhik, V. M. (1972). *Theory of Unsteady Filtration of Fluids and Gases*, Nedra, Moscow.
Barrero, A., and Sanmartin, J. R. (1977). *Phys. Fluids* **20,** 1155.
Bear, J. (1982). *Dynamics of Fluids in Porous Media*, American Elsevier, New York.
Bertschinger, E.W. (1985a). *Ap. J. Supp.*, **58,** 1.
Bertschinger, E.W. (1985b). *Ap. J. Supp.*, **58,** 39.
Birkhoff, G. (1960). *Hydrodynamics, a Study in Logic, Fact, and Similitude*. 2nd Ed. Princeton Univ. Press.

Blasius, H. (1908). *Z. Math. Phys.*, **56,** 37.
Bluman, G. W., and Cole, J. D. (1974). *Similarity Methods for Differential Equations*, Springer-Verlag, New York.
Bluman, G. W., and Reid, G. J. (1988). *IMA J. Appl. Math.*, **40,** 87.
Bluman, G. W., Kumei, S., and Reid, G. J. (1988). *J. Math. Phys.* **29,** 806
Book, D. L., and Bernstein, I. B. (1979). *Phys. Fluids*, **22,** 79.
Boyer, R. H. (1962). *J. of Math. and Phys.*, **40,** 41.
Britter, R. E. (1979). *Atmos. Environ.*, **13,** 1241.
Brooke Benjamin, T. (1968). *J. Fluid Mech.*, **31,** 209.
Bruhat, G. (1963). *Cours de Phisique Générale*, Masson & Cie., Paris.
Brushlinski, K. V., and Kazhdan, Ya. M. (1963). *Uspekhi Mat. Nauk* **18(2),** 3.
Buckmaster, J. (1977). *J. Fluid Mech.*, **81,** 735.
Caporaso, G. J. (1982). *Phys. Fluids* **25,** 436.
Coggeshall, S. V., and Axford, R. A. (1986). *Phys. Fluids* **29,** 2398.
Courant, R., and Friedrichs, K. O. (1948). *Supersonic Flow and Shock Waves*, Interscience, New York.
Didden, N., and Maxworthy, T. (1982). *J. Fluid Mech.*, **121,** 27.
Diez, J. (1989). In preparation.
Diez, J. (1990). In preparation.
Diez, J., Gratton, J., and Minotti, F. (1989). ICTP, IC/89/70.
Diez, J., Gratton, R., and Gratton, J. (1990). *Anales Asociación Fisica Argentina*, AFA, in press.
Fannelop, T. K., and Waldman, G. D. (1972). *AIAA J.*, **10,** 506.
Felber, F. S., Liberman, M. A., and Velikovich, A. L. (1988a). *Phys. Fluids* **31,** 3675.
Felber, F. S., Liberman, M. A., and Velikovich, A. L. (1988b). *Phys. Fluids* **31,** 3683.
Ferro Fontán, C., Gratton, J., and Gratton, R. (1975). *Phys. Lett.*, **55A,** 35.
Ferro Fontán, C., Gratton, J., and Gratton, R. (1977). *Nucl. Fusion* **17,** 135.
Fujimoto, Y., and Mishkin, E. A. (1978a). *J. Fluid Mech.*, **89,** 61.
Fujimoto, Y., and Mishkin, E. A. (1978b). *Phys. Fluids* **21,** 1933.
Gaffet, B. (1983). *J. Fluid Mech.*, **134,** 179.
Gaffet, B. (1985). *J. Math. Phys.*, **26,** 2165.
Gratton, J., and Minotti, F. (1988). Meeting of the AFA, Mar del Plata, to be published.
Gratton, J., and Minotti, F. (1990). *J. Fluid Mech.*, **210,** 155.
Gratton, J., Mahajan, S. M., and Minotti, F. (1988). ICTP, IC/88/370.
Grundy, R. E., and Rottman, J. W. (1985). *J. Fluid Mech.*, **156,** 39.
Grundy, R. E., and Rottman, J. W. (1986). *J. Fluid Mech.*, **169,** 337.
Guderley, G. (1942). *Luftfahrtforschung*, **19,** 302.
Henriksen, R. N., and Wesson, P. S. (1978). *Astrophys. Space Sci.*, **53,** 429.
Hoult, D. P. (1972). *Ann. Rev. Fluid Mech.*, **4,** 341.
Hunter, C. (1960). *J. Fluid Mech.*, **8,** 241.
Hunter, C. (1963). *J. Fluid Mech.*, **15,** 289.
Huppert, H. E., and Simpson, J. E. (1980). *J. Fluid Mech.*, **90,** 785.
Huppert, H. E. (1982). *J. Fluid Mech.*, **121,** 43.
Huppert, H. E. (1986). *J. Fluid Mech.*, **173,** 557.
Kamenomostskaya, S. L. (1957). *Dokl. Akad. Nauk SSSR* **116(1),** 18.
Kath, W. L., and Cohen, D. S. (1982). *Studies in Appl. Math.*, **67,** 79.
Kerr, R. C., and Lister, J. R. (1987). *Earth Planet. Sci. Lett.*, **85,** 241.
Kurth, R. (1972). *Dimensional Analysis and Group Theory in Astrophysics*, Pergamon Press, New York.
Lacey, A. A., Ockendon, J. R., and Tayler, A. B. (1982). *J. Appl. Math.*, **42,** 1252.
Landau, L. D., and Lifschitz, E. M. (1959a). *Mechanics*, Pergamon Press, New York.
Landau, L. D., and Lifschitz, E. M. (1959b) *Fluid Mechanics*, Pergamon Press, New York.

Lazarus, R. B., and Richtmeyer, R. D. (1977). Report LA-6823-MS, Los Alamos Sc. Lab.
Lengyel, L. L. (1973). *AIAA J.*, **11**, 1347.
Li, W. H., and Lam, S. H. (1964). *Principles of Fluid Mechanics*, Addison Wesley, New York.
Liberman, M. A., and Velikovich, A. L. (1986). *Nucl. Fusion* **26**, 709.
Liberman, M. A., and Velikovich, A. L. (1989). *Phys. Fluids* **B1**, 1271.
Lord Rayleigh (1917). *Phil. Mag.*, **34**, 94.
Maxworthy, T. (1983). *J. Fluid Mech.*, **128**, 247.
Meyer-ter-Vehn, J. and Schalk, C. (1982). *Z. Naturforsch.*, **37a**, 955.
Migdal, A. B. (1977). *Qualitative Methods in Quantum Theory*, Benjamin, New York.
Morgan, A. J. A. (1952). *Q. J. Math.*, **3**, 250.
Nakamura, Y. (1983). *Phys. Fluids* **26**, 1234.
Ostriker, J. P., and McKee, C. F. (1988). *Rev. Modern Phys.*, **60**, 1.
Ovsyannikov, L. V. (1962). *Group properties of Differential Equations*, Izd-vo SO Akad. Nauk SSSR, Novosibirsk.
Pakula, R. E., and Sigel, R. (1985). *Phys. Fluids* **26**, 232.
Pattle, R. E. (1959). *Q. J. Mech. Appl. Math.*, **12**, 407.
Peletier, L. A. (1981). *Applications of Nonlinear Analysis in the Physical Sciences*, Pitman Adv. Pub. Prog., Boston.
Prandtl, L. (1904). *Int. Congr. Math.*, Heidelberg, 484.
Rodriguez, M., and Linan, A. (1978). Report J.E.N. 405, Junta de Energía Nuclear, Madrid.
Rottman, J. W., and Simpson, J. E. (1983). *J. Fluid Mech.*, **135**, 95.
Sanmartin, J. R., and Barrero, A. (1978a). *Phys. Fluids* **21**, 1957.
Sanmartin, J. R., and Barrero, A. (1978b). *Phys. Fluids* **21**, 1967.
Sanyal, A. K., Bannerjee, A., and Ray, D. (1985). *J. Math. Phys.*, **26**, 2693.
Schmalz, R. F. (1986). *Phys. Fluids* **29**, 1389.
Sedov, L. I. (1946). *Prikl. Mat. Mekh.*, **10**, 241, No. 2.
Sedov, L. I. (1959). *Similarity and Dimensional Methods in Mechanics*, Academic Press, New York.
Seshadri, R., and Na, T. Y. (1985). *Group Invariance in Engineering Boundary Problems*, Springer-Verlag, New York.
Simpson, J. E., and Britter, R. E. (1979). *J. Fluid Mech.*, **94**, 477.
Simpson, J. E. (1982). *Ann. Rev. Fluid Mech.*, **14**, 341.
Smyth, N. F., and Hill, J. M. (1988). *IMA J. Appl. Math.*, **40**, 73.
Stanyukovich, K. P. (1960). *Unsteady Motion of Continuous Media*, Pergamon, London.
Tappert, F. D. (1977). *Proc. Symp. Appl. Math.*, **21**, 55.
Taylor, G. I. (1950). *Proc. R. Soc. London, Ser. A* **201**, 175.
Thomas, L. P., Pais, V., Gratton, R., and Diez, J. (1986). *Phys. Fluids* **29**, 676.
Töpfer, C. (1912). *Z. Maths. Phys.*, **60**, 397.
Velikovich, A. L., Gol'berg, S. M., Liberman, M. A., and Felber, F. S. (1985). *Sov. Phys. JETP* **61**, 261.
Von Kármán, T. (1940). *Bull. Am. Math. Soc.*, **46**, 615.
Welsh, R. L. (1967). *J. Fluid Mech.*, **29**, 61.
Whitham, G. B. (1974). *Linear and Nonlinear Waves*, Wiley, New York.
Yih, C.-S. (1965). *Dynamics of non Homogeneous Fluids*, Macmillan, New York.
Yousaf, M. (1986). *Phys. Fluids* **29**, 680.
Zel'dovich, Ya. B., and Raizer, Yu. P. (1967). *Physics of Shock Waves and High Temperature Hydrodynamics Phenomena*, Academic Press, New York.
Zel'dovich, Ya. B. (1956). *Akust. Zh.*, **6(1)**, 28.
Zmitrenko, N. V., and Kurdyumov, S. P. (1975). *Sov. Phys. Dokl.*, **19**, 660.

NOTES FOR CONTRIBUTORS

TYPESCRIPTS

Please consult the editors before submission of an article, since the articles are generally solicited. Papers should be typed with double spacing on good quality paper and submitted in triplicate to Dr. V. M. Canuto, NASA, Institute for Space Studies, 2880 Broadway, New York, N.Y. 10025 or to Dr. Bruce G. Elmegreen, Department of Astronomy, Columbia University, New York, N.Y. 10027. Submission of an article to this journal will be taken to imply that it represents work not previously published, that it is not being considered elsewhere for publication, and that if accepted for publication will not be published elsewhere in the same form, in any language, without the consent of editor and publisher. It is a condition of the acceptance by the editor of a typescript for publication that the publishers acquire automatically the copyright in the typescript throughout the world.

FIGURES

All figures should be numbered with consecutive arabic numerals, have descriptive captions, and be mentioned in the text. An approximate position for each figure should be indicated in the margin.
Preparation: Figures submitted must be of a high enough standard for direct reproduction. Line drawings should be prepared in black (india) ink on white paper or tracing cloth, with all lettering and symbols included. Alternatively, good sharp photoprints ("glossies") are acceptable. Photographs intended for halftone reproduction must be good glossy original prints, of maximum contrast. Each figure should be clearly labelled with the author's name and figure number. Redrawing or retouching of unusable figures will be charged to authors.
Size: Figures should be planned so that they reduce to 10½cm column width. The *preferred* width of submitted line drawings is 16 to 21 cm, with capital lettering 4 mm high, for reduction by one-half. Photographs for halftone reproduction should be approximately twice the desired size.
Captions: A list of figure captions should be typed on a separate sheet and included with the typescript.

COLOR PLATES

Whenever the use of color is an integral part of the research, or where the work is generated in color, the journal will publish the color illustrations without charge to the authors. Reprints in color will carry a surcharge. Please write to the Editor for details.

EQUATIONS

Wherever possible, mathematical equations should be typewritten, with subscripts and superscripts clearly shown. It is helpful to identify unusual or ambiguous symbols in the margin when they first occur. To simplify typesetting, please use fractional exponents instead of root signs, and use the solidus (/) in simple fractions.
Marking: The printer normally sets mathematical symbols in italics, except for obvious groups like sin and log. Any symbols which are to be left in roman (upright) type should be encircled in pencil in the typescript: bold symbols should be underlined with a wavy line.

TABLES

Number tables consecutively with roman numerals, and give each a clear descriptive caption. Avoid the use of vertical rules in tables.

REFERENCES

References are indicated in the text by the Harvard (name and date) system: Either "Recent work (Smith, 1970) . . ." or "Recently Smith (1970) has found . . ." The full list of references should be collected at the end of the paper in alphabetical order. Listed references should include all details, but excluding article titles; abbreviations of journal titles should follow the usage in *Physics Abstracts*. The initials of the first-named author should be placed after his name, the year of publication follows the last-named author. Examples:
Smith, L. F., and L. H. Aller (1971). *Astrophys. J.*, **164**, 275.
Hubbard, W. B. (1969). In S. S. Kumar (Ed.), *Low Luminosity Stars*, Gordon and Breach, New York.

PROOFS

Authors will receive page proofs (including figures) by airmail for correction, which must be returned to the printer within 48 hours of receipt. Please ensure that a full postal address is given on the first page of the typescript, so that proofs are not delayed in the post. Authors' alterations in excess of 10% of the original composition cost will be charged to authors.